

# Affine and finite Lie algebras and integrable Toda field equations on discrete space-time

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## Abstract

Difference-difference systems are suggested corresponding to the Cartan matrices of any simple or affine Lie algebra. In the cases of the algebras  $A_N, B_N, C_N, G_2, D_3, A_1^{(1)}, A_2^{(2)}, D_N^{(2)}$  these systems are proved to be integrable. For the systems corresponding to the algebras  $A_2, A_1^{(1)}, A_2^{(2)}$  generalized symmetries are found. For the systems  $A_N, B_N, C_N, G_2, D_3$  complete sets of integrals are found. The Lax representation for the difference-difference systems corresponding to  $A_N, B_N, C_N, A_1^{(1)}, D_N^{(2)}$  are presented.

*Keywords:* affine Lie algebra, difference-difference systems, S-integrability, Darboux integrability, Toda field theory, integral, symmetry, Lax pair.

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# 1 Introduction

Systems of partial differential equations of the form

$$r_{x,y}^i = e^{\sum_{j=1}^{j=N} a_{ij} r^j}, \quad i = 1, 2, \dots, N, \quad (1.1)$$

called generalized two-dimensional Toda lattices have very important applications in Liouville and conformal field theories, they are studied in details (see [1]-[12] and the references therein). Here the matrix  $A = \{a_{i,j}\}$  is the Cartan matrix of an arbitrary finite or affine Lie algebra. It is known that in the former case system (1.1) is Darboux integrable while in the latter case – S-integrable. The widely known Drinfel'd-Sokolov formalism allows one to construct the Lax representation for the system (1.1) in terms of the Lie algebra canonically associated with the corresponding Cartan matrix  $A$ .

The problem of finding discrete versions of the system is intensively studied (see, for instance, [13]-[26]). Recently, in [27] integrable differential-difference analog of system (1.1) was suggested

$$v_{1,x}^i - v_x^i = e^{\sum_{j=1}^{j=i-1} a_{i,j} v^j + \sum_{j=i+1}^{j=N} a_{i,j} v_1^j + \frac{1}{2} a_{i,i} (v^i + v_1^i)}, \quad i = 1, 2, \dots, N. \quad (1.2)$$

Here the functions  $v^j = v^j(n, x)$ ,  $j = 1, \dots, N$  are the searched field variables. The subindex denotes a shift of the discrete variable  $n$  or the derivative with respect to  $x$ :  $r_k^j = r^j(n + k, x)$  and  $r_x^j = \frac{\partial}{\partial x} r^j(n, x)$ . A new example of the semi-discrete system corresponding to  $C_N$  was recently studied in [28].

In the present article we study the problem of further discretization of system (1.1), i.e. the problem of finding a rule allowing to assign to any Cartan matrix a system of integrable difference-difference equations approximating in the continuum limit system (1.2) and therefore system (1.1). As it was pointed out in [14] the problem of discretization is important from physical viewpoint, they might have applications in discrete field theory and in quantum physics (see also [15]-[17]). They can also be regarded as difference schemes in numerical computations (see [31]).

Following questions were addressed in [14]:

- 1) whether there exists an integrable discrete version for any two dimensional Toda field equation (1.1);
- 2) which kind of algebraic structure (like Lie algebra or Lie group) is naturally related to discrete versions.

The subject has intensively been studied during the last 10-15 years. Various discrete versions of Toda field equations were investigated in literature (see [16], [17], [18], [19], [21] and references therein). As an alternative answer to the first question we suggest a fully discrete version of system (1.1) in the following form:

$$e^{-u_{1,1}^i + u_{1,0}^i + u_{0,1}^i - u_{0,0}^i} - 1 = e^{\sum_{j=1}^{j=i-1} a_{i,j} u_{0,1}^j + \sum_{j=i+1}^{j=N} a_{i,j} u_{1,0}^j + \frac{1}{2} a_{i,i} (u_{0,1}^i + u_{1,0}^i)}, \quad i = 1, 2, \dots, N, \quad (1.3)$$

which evidently approximates (1.1) and (1.2). Here  $u^j = u^j(n, m)$ ,  $j = 1, 2, \dots, N$ , is a set of the field variables. The subindex indicates shifts of the arguments  $n, m$  as follows  $u_{i,k}^j := u^j(n + i, m + k)$ .

The main result of the present article is in the formulating the conjecture below and proving it for numerous examples of Lie algebras.

**Conjecture.** a) If  $A$  is the Cartan matrix of a semi-simple Lie algebra then (1.3) is Darboux integrable, in other words, it admits a complete set of integrals in both directions. Roughly speaking Darboux integrable difference-difference equations can be reduced to ordinary difference equations.

b) If  $A$  is the Cartan matrix of an affine Lie algebra then (1.3) is  $S$ -integrable (it admits exact multi-soliton solutions, generalized symmetries, Lax pairs, etc.).

Recall that a function  $F$  (function  $G$ ) is called an  $m$ -integral (respectively an  $n$ -integral) if the following identity holds  $D_m F = F$  (or,  $D_n G = G$ ), where the shift operators  $D_m, D_n$  act due to the rules  $D_m f(n, m) = f(n, m + 1)$ ,  $D_n f(n, m) = f(n + 1, m)$ . Equations admitting a complete set of integrals in both directions are called Darboux integrable.

Part a) of the conjecture is proved for the cases:  $A_N, B_N, C_N, G_2, D_3$  by finding complete sets of integrals. In the cases  $A_2, B_2, G_2$  the exponential system (1.3) reads as follows

$$\begin{aligned} e^{-u_{1,1}^1 + u_{1,0}^1 + u_{0,1}^1 - u_{0,0}^1} - 1 &= e^{u_{0,1}^1 + u_{1,0}^1 - u_{1,0}^2}, \\ e^{-u_{1,1}^2 + u_{1,0}^2 + u_{0,1}^2 - u_{0,0}^2} - 1 &= e^{-cu_{0,1}^1 + u_{0,1}^2 + u_{1,0}^2}, \end{aligned} \quad (1.4)$$

where  $c$  takes only three values  $c = 1, 2, 3$ . Here the values  $c = 1, c = 2, c = 3$  correspond to the algebras  $A_2, B_2, G_2$  respectively. Let us introduce a notation for the nonlinear second order difference operator by setting

$$\Delta(u) := e^{-u_{1,1} + u_{1,0} + u_{0,1} - u_{0,0}} - 1, \quad (1.5)$$

then the last system takes a compact form

$$\begin{aligned}\Delta(u^1) &= e^{u_{0,1}^1 + u_{1,0}^1 - u_{1,0}^2}, \\ \Delta(u^2) &= e^{-cu_{0,1}^1 + u_{0,1}^2 + u_{1,0}^2}.\end{aligned}\tag{1.6}$$

Integrals in both directions for system (1.6) are given in Section 2.

Since the map defined in (1.3) converts any  $N \times N$  matrix to a system of difference-difference equations one can easily specify the form of systems corresponding to any Cartan matrix canonically related to a finite or affine Lie algebra both of classical series or exceptional. For instance, for the algebra  $D_3$  with the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix},\tag{1.7}$$

we have an integrable system of the form

$$\begin{aligned}\Delta(u^1) &= e^{u_{0,1}^1 + u_{1,0}^1 - u_{1,0}^2 - u_{1,0}^3}, \\ \Delta(u^2) &= e^{-u_{0,1}^1 + u_{0,1}^2 + u_{1,0}^2}, \\ \Delta(u^3) &= e^{-u_{0,1}^1 + u_{0,1}^3 + u_{1,0}^3}.\end{aligned}\tag{1.8}$$

Complete sets of integrals for this system is given in Section 2.

For the Kac-Moody algebra  $A_2^{(2)}$  with the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\tag{1.9}$$

the corresponding system is

$$\begin{aligned}\Delta(u^1) &= e^{u_{0,1}^1 + u_{1,0}^1 - u_{1,0}^2}, \\ \Delta(u^2) &= e^{-4u_{0,1}^1 + u_{0,1}^2 + u_{1,0}^2}.\end{aligned}\tag{1.10}$$

For the difference-difference systems corresponding the algebras  $A_2$ ,  $A_1^{(1)}$ ,  $A_2^{(2)}$  generalized symmetries are found, for  $A_2$  the symmetries are local, for  $A_1^{(1)}$ ,  $A_2^{(2)}$  they are nonlocal. In the

literature generalized symmetries are regarded as a criterion of integrability. Thus one concludes that systems (1.10), (3.1) provide new examples of  $2 \times 2$  S-integrable quad graph models.

Stress that the number of the field variables in all three systems (1.1), (1.2), (1.3) coincides with the rank of the corresponding Lie algebra.

It is known that any system of difference-difference equations of hyperbolic type admits a pair of characteristic Lie algebras which are effectively evaluated (see [24]). Characteristic Lie algebras of system (1.3) are closely connected with the Lie algebra canonically related to the given Cartan matrix  $A$ . In our opinion this relation could allow one to answer the second question in [14] (see the list of questions above).

The problem of developing discrete versions of the Drinfel'd-Sokolov formalism is also challenging since today systems of discrete equations are very popular. They have a large variety of application in theoretical physics, in discrete geometry, in the theory of tau functions in lattice Toda field equations. The present article would also provide an "experimental" background for creating a discrete theory parallel to the Drinfel'd-Sokolov formalism.

The article is organized as follows. In Section 2 a method of discretization of Darboux integrable models preserving integrability is discussed. The method is explained in details by the example of the Liouville equation. Then it is applied to the system (1.2) to get its difference-difference analog. The case  $B_2$  is studied in more details. These simulations allowed us to guess the formula (1.3). Complete sets of integrals for the systems  $A_2$ ,  $B_2$ ,  $C_2$ ,  $G_2$ ,  $D_3$  are presented.

Generalized symmetries are evaluated for the systems corresponding  $A_2$ ,  $A_1^{(1)}$ ,  $A_2^{(2)}$  in Section 3. In Section 4 the concept of the characteristic Lie algebra (see [24]) of discrete models is briefly discussed. Here a complete description of such algebra is obtained for the system of difference-difference equations  $A_2$ .

In the fifth Section cutting off conditions for the Hirota equation preserving integrability are studied by using the method suggested in [21]. It turned out that some of the systems in the class (1.3) can be obtained from the Hirota equation by imposing proper boundary conditions. Since the cutting off conditions are compatible with the Lax representation one can derive together with the reduced discrete system also its Lax pair. In §5 the Lax pairs for the difference-difference systems  $A_N$ ,  $B_N$ ,  $C_N$ ,  $A_1^{(1)}$ ,  $D_N^{(2)}$  are found.

In §6 an algorithm is suggested to look for integrals via the Lax pair. In §7 widely known periodical reduction of the Hirota equation is discussed. The Lax pair for this system found in [32] is rewritten in terms of the Cartan-Weyl basis for the algebra  $A_N^{(1)}$ . Emphasize that the difference-difference system obtained as a periodical reduction and that given by the formula (1.3) corresponding to  $A_N^{(1)}$  are not equivalent (see Remark 3 below) .

## 2 Systematic approach to the problem of discretization of the Darboux integrable systems

The problem of finding integrable discretizations of the integrable partial differential equation is very complicated and not enough studied. The same is true for evaluating the continuum limit for discrete models. In [22] an effective algorithm of discretization (as well as evaluation of the continuum limit) of Darboux integrable equations is suggested based on the integrals. In this section we discuss the essence of the algorithm and apply it to exponential type systems.

### 2.1 Explanation of the method with example of Liouville equation

It is well-known that the famous Liouville equation  $u_{xy} = e^u$  admits integrals in both directions:  $W = u_{xx} - 0.5u_x^2$  and  $\bar{W} = u_{yy} - 0.5u_y^2$ . Indeed, it is very easy to check that  $D_y W = 0$  and  $D_x \bar{W} = 0$ .

Consider now the problem of finding all chains of the form

$$t(n+1, x) = f(x, t(n, x), t(n+1, x), t_x(n, x)), \quad (2.1)$$

having the function  $I = t_{xx} - \frac{1}{2}t_x^2$  as their  $n$ -integral. Equality  $D_n I = I$  implies

$$f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx} - \frac{1}{2}f^2 = t_{xx} - \frac{1}{2}t_x^2. \quad (2.2)$$

By comparing the coefficients before  $t_{xx}$  in (2.2) we have  $f_{t_x} = 1$ . Therefore,

$$f(x, t, t_1, t_x) = t_x + d(x, t, t_1). \quad (2.3)$$

We substitute (2.3) into (2.2) and get  $d_x + d_t t_x + d_{t_1} t_x + d_{t_1} d - \frac{1}{2}t_x^2 - dt_x - \frac{1}{2}d^2 = -\frac{1}{2}t_x^2$ , or equivalently,  $d_t + d_{t_1} - d = 0$  and  $d_x + d_{t_1} d - \frac{1}{2}d^2 = 0$ . We solve the last two equations

simultaneously and find that  $d = e^{t_1} K(x, t_1 - t)$ , where  $K = Ce^{-\frac{1}{2}(t_1 - t)}$  and  $C$  is an arbitrary constant. Therefore, chain (2.1) with  $n$ -integral  $I = t_{xx} - \frac{1}{2}t_x^2$  becomes  $t_{1x} = t_x + Ce^{(t_1 + t)/2}$ . The last equation in its turn admits also an  $x$ -integral  $F = e^{(t_1 - t)/2} + e^{(t_1 - t_2)/2}$ .

The next step consists in describing equations of the form

$$v(n+1, m+1) = f(v(n, m), v(n+1, m), v(n, m+1)) \quad (2.4)$$

with  $m$ -integral  $\bar{I} = e^{v_1 - v} + e^{v_1 - v_2}$ . Denote  $w_j := e^{-v_j}$ ,  $j = 0, 1, 2$ , and  $\bar{w}_1 := e^{-\bar{v}_1}$ . In the new variables  $\bar{I} = \frac{w+w_2}{w_1}$  is an  $m$ -integral of equation  $w_{1,1} = g(w, w_1, \bar{w}_1)$ .  $D_m \bar{I} = \bar{I}$  implies

$$\frac{w_2 + w}{w_1} = \frac{g_1 + \bar{w}_1}{g}. \quad (2.5)$$

We differentiate both sides of (2.5) with respect to  $w_2$  and apply the shift operator  $D_n^{-1}$ , we have

$$\frac{1}{w_1} = \frac{g_{1w_2}}{g} \Rightarrow D_n^{-1} \left( \frac{1}{w_1} \right) = D_n^{-1} \left( \frac{g_{1w_2}}{g} \right) \Rightarrow g_{w_1} = \frac{\bar{w}_1}{w}.$$

Therefore,

$$g = \frac{\bar{w}_1 w_1}{w} + c(w, \bar{w}_1), \quad g_1 = \frac{g w_2}{w_1} + c(w_1, g). \quad (2.6)$$

We substitute (2.6) into (2.5) and get

$$g \frac{w}{w_1} = c(w_1, g) + \bar{w}_1. \quad (2.7)$$

Substitution of (2.6) into (2.7) implies that  $c(w, \bar{w}_1)w = c(w_1, g)w_1$ , or the same,  $c(w, \bar{w}_1)w = D_n(c(w, \bar{w}_1)w)$ . Suppose that equation  $w_{1,1} = g(w, w_1, \bar{w}_1)$  does not admit an  $m$ -integral of the first order, then  $c(w, \bar{w}_1)w = D_n(c(w, \bar{w}_1)w) = C = \text{const}$ . Thus,  $c(w, \bar{w}_1) = C/w$ . Finally,  $g(w, w_1, \bar{w}_1) = \frac{\bar{w}_1 w_1}{w} + Cw^{-1}$ . Therefore, the equations (2.4) with  $m$ -integral  $\bar{I} = e^{v_1 - v} + e^{v_1 - v_2}$  becomes  $e^{-v_{1,1} - v} = C + e^{-v_1 - \bar{v}_1}$ , where  $C$  is an arbitrary constant. Note that this equation is symmetric with respect to variables  $v_1$  and  $\bar{v}_1$ . Therefore,  $n$ -integral for the equation can be obtained by simply changing in  $m$ -integral variables  $v_j$  into variables  $\bar{v}_j$ ,  $j = 1, 2$ .



## 2.2 Application of the algorithm of discretization to the systems of series $B_2$

Apply the reasonings above to the system (1.1) corresponding to the algebra  $B_2$ . The first step has already been done in [27], where the differential-difference system was found:

$$\begin{aligned} u_{1,x}^1 - u_x^1 &= e^{u^1+u_1^1-u_1^2}, \\ u_{1,x}^2 - u_x^2 &= e^{-2u^1+u^2+u_1^2}, \end{aligned} \quad (2.8)$$

admitting two independent  $n$ -integrals having the form:

$$\begin{aligned} I_1 &= 2u_{xx}^1 + u_{xx}^2 - 2(u_x^1)^2 + 2u_x^1 u_x^2 - (u_{xx}^2)^2, \\ I_2 &= u_{[4]}^1 + u_x^1(u_{[3]}^2 - 2u_{[3]}^1) + u_{xx}^1(4u_x^1 u_x^2 - 2(u_x^1)^2 - (u_x^2)^2) + \\ &\quad + u_{xx}^1(u_{xx}^2 - u_{xx}^1) + u_{xx}^2 u_x^1(u_x^1 - 2u_x^2) + (u_x^1)^4 + (u_x^1)^2(u_x^2)^2 - 2(u_x^1)^2 u_x^2. \end{aligned}$$

It was shown in [27] that system (2.8) admits also  $x$ -integrals:

$$\begin{aligned} F_1 &= e^{-u_0^1+u_1^1} + e^{-u_1^1+u_2^1+u_2^2-u_3^2} + e^{u_1^1-u_2^1-u_1^2+u_2^2} + e^{u_2^1-u_3^1}, \\ F_2 &= e^{-u_0^2+u_1^2} + e^{-2u_0^1+2u_1^1+u_1^2-u_2^2} + 2e^{-u_0^1+2u_1^1-u_2^1} + e^{2u_1^1-2u_2^1-u_1^2+u_2^2} + e^{u_2^2-u_3^2}. \end{aligned} \quad (2.9)$$

The goal of this section is to construct a difference-difference system having the same  $n$ -integrals  $F_1$  and  $F_2$ .

To make the formulas shorter we change the variables:

$$a = e^{-u^1}, \quad b = e^{-u^2} \quad (2.10)$$

where  $a, b$  are new unknowns.

The given  $n$ -integrals in these variables read:

$$\begin{aligned} F_1 &= \frac{a_0}{a_1} + \frac{a_1 b_3}{a_2 b_2} + \frac{a_2 b_1}{a_1 b_2} + \frac{a_3}{a_2}, \\ F_2 &= \frac{b_0}{b_1} + \frac{a_0^2 b_2}{a_1^2 b_1} + 2 \frac{a_0 a_2}{a_1^2} + \frac{a_2^2 b_1}{a_1^2 b_2} + \frac{b_3}{b_2}. \end{aligned} \quad (2.11)$$

Here and below we omit the second index, because its value is zero for all considered variables.

Now we will look for the equations desired in the variables  $a, b$  in the form:

$$\begin{aligned} a_{1,1} &= f(a_{0,0}, a_{1,0}, a_{0,1}, b_{0,0}, b_{1,0}, b_{0,1}), \\ b_{1,1} &= g(a_{0,0}, a_{1,0}, a_{0,1}, b_{0,0}, b_{1,0}, b_{0,1}). \end{aligned} \quad (2.12)$$

Substitute the integrals (2.11) into the equations for  $n$ -integrals:  $D_m F_1 = F_1$ ,  $D_m F_2 = F_2$  and bring them to the following form:

$$\frac{a_{0,1}}{f} + \frac{f D_n^2 g}{D_n f D_n g} + \frac{g D_n f}{f D_n g} + \frac{D_n^2 f}{D_n f} = \frac{a_0}{a_1} + \frac{a_1 b_3}{a_2 b_2} + \frac{a_2 b_1}{a_1 b_2} + \frac{a_3}{a_2}, \quad (2.13a)$$

$$\frac{b_{0,1}}{g} + \frac{a_{0,1}^2 D_n g}{f^2 g} + 2 \frac{a_{0,1} D_n f}{f^2} + \frac{g D_n f^2}{f^2 D_n g} + \frac{D_n^2 g}{D_n g} = \frac{b_0}{b_1} + \frac{a_0^2 b_2}{a_1^2 b_1} + 2 \frac{a_0 a_2}{a_1^2} + \frac{a_2^2 b_1}{a_1^2 b_2} + \frac{b_3}{b_2}. \quad (2.13b)$$

By differentiating equation (2.13b) with respect to the variables  $a_3, b_3$  we get equations

$$\begin{aligned} \frac{1}{D_n g} D_n^2 \frac{\partial g}{\partial a_1} &= 0 \Leftrightarrow \frac{\partial g}{\partial a_1} = 0, \\ \frac{1}{D_n g} D_n^2 \frac{\partial g}{\partial b_1} &= \frac{1}{b_2} \Leftrightarrow \frac{\partial g}{\partial b_1} = \frac{b_{0,1}}{b_0}, \end{aligned}$$

which imply immediately:

$$g(a_{0,0}, a_{1,0}, a_{0,1}, b_{0,0}, b_{1,0}, b_{0,1}) = \frac{b_{1,0} b_{0,1}}{b_{0,0}} + \frac{g_1(a_{0,0}, a_{0,1}, b_{0,0}, b_{0,1})}{b_{0,0}}.$$

Similarly, differentiation of (2.13a) with respect to  $a_3, b_3$  yields

$$\begin{aligned} \frac{1}{D_n f} D_n^2 \frac{\partial f}{\partial a_1} &= \frac{1}{a_2} \Leftrightarrow \frac{\partial f}{\partial a_1} = \frac{a_{0,1}}{a_0}, \\ \frac{f D_n g}{b_2 D_n f D_n g} + \frac{1}{D_n f} D_n^2 \frac{\partial f}{\partial b_1} &= \frac{a_1}{a_2 b_2} \Leftrightarrow \frac{\partial f}{\partial b_1} = \frac{1}{a_0 b_0} D_n^{-1}(a_0 f - a_1 a_{0,1}). \end{aligned}$$

From these equations we get:

$$f(a_{0,0}, a_{1,0}, a_{0,1}, b_{0,0}, b_{1,0}, b_{0,1}) = \frac{a_{1,0} a_{0,1}}{a_{0,0}} + \frac{C b_{1,0}}{a_{0,0}}.$$

By differentiating (2.13) with respect to  $a_2$  we obtain

$$\begin{aligned} \frac{a_{1,1}}{b_2 b_{2,1}} \frac{\partial}{\partial a_2} \frac{D_n^2 g_1}{a_{2,1}} + \frac{b_{1,1}}{a_1 b_{2,1}} &= \frac{b_1}{a_1 b_2}, \\ 2 \frac{a_{0,1}}{a_{1,1} a_1} + 2 \frac{a_{2,1} b_{1,1}}{a_1 a_{1,1} b_{2,1}} + \frac{1}{b_{2,1} b_{2,0}} \frac{\partial}{\partial a_2} D_n^2 g_1 &= 2 \frac{a_0}{a_1} + 2 \frac{a_2 b_1}{a_1^2 b_2}. \end{aligned}$$

The last two equations can be rewritten as follows:

$$\begin{aligned} \frac{\partial}{\partial a_1} \frac{D_n g_1}{a_{1,1}} &= \frac{g_1}{a_0 a_{0,1}}, \\ \frac{\partial}{\partial a_1} D_n g_1 &= 2 \frac{a_{1,1} g_1}{a_{0,1} a_0}. \end{aligned}$$

By getting rid of the variable  $\partial_{a_1} D_n g_1$  in these equations we find

$$\frac{D_n g_1}{a_{1,1}^2} = \frac{g_1}{a_{0,1}^2} \Leftrightarrow g_1 = C_1 a_{0,1}^2.$$

The last implication is due to the assumption that  $F_1$  and  $F_2$  are integrals of the lowest order.

As a result we get equations:

$$a_{1,1}a_{0,0} - a_{1,0}a_{0,1} = Cb_{1,0},$$

$$b_{1,1}b_{0,0} - b_{1,0}b_{0,1} = C_1 a_{0,1}^2.$$

After some rescaling we can put  $C = C_1 = 1$ . Finally, we end up with the system of discrete equations

$$a_{1,1}a_{0,0} - a_{1,0}a_{0,1} = b_{1,0},$$

$$b_{1,1}b_{0,0} - b_{1,0}b_{0,1} = a_{0,1}^2.$$

corresponding to the algebra  $B_2$ . Turn back to the original variables  $u^1, u^2$  (see (2.10))

$$\begin{aligned} \Delta u^1 &= e^{u_{0,1}^1 + u_{1,0}^1 - u_{1,0}^2}, \\ \Delta u^2 &= e^{-2u_{0,1}^1 + u_{0,1}^2 + u_{1,0}^2}. \end{aligned} \tag{2.14}$$

By construction the last system admits pair of integrals (2.9) in which the second index for the variables  $u^1, u^2$  is omitted, since its value is the same for variables. To find the integrals in the other direction one uses the discrete symmetry  $n \rightarrow -m, m \rightarrow -n$  of the system.

## 2.3 Integrals of the systems corresponding to the series $A_2, G_2, D_3$

### 2.3.1 System corresponding to the algebra $A_2$

The Cartan matrix of  $A_2$  is:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The discrete system for  $A_2$  looks as follows:

$$\begin{aligned} \Delta(u^1) &= e^{u_{0,1}^1 + u_{1,0}^1 - u_{1,0}^2}, \\ \Delta(u^2) &= e^{-u_{0,1}^1 + u_{0,1}^2 + u_{1,0}^2}. \end{aligned} \tag{2.15}$$

For system (2.15)  $m$ -integrals are

$$\begin{aligned} F_1 &= e^{-u_{0,0}^2+u_{1,0}^2} + e^{-u_{0,0}^1+u_{1,0}^1+u_{1,0}^2-u_{2,0}^2} + e^{u_{1,0}^1-u_{2,0}^1}, \\ F_2 &= e^{-u_{0,0}^1+u_{1,0}^1} + e^{u_{1,0}^1-u_{2,0}^1-u_{1,0}^2+u_{2,0}^2} + e^{u_{2,0}^2-u_{3,0}^2}. \end{aligned}$$

### 2.3.2 System corresponding to the algebra $G_2$

The Cartan matrix of  $G_2$  is:

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The discrete system for  $G_2$  is of the form:

$$\begin{aligned} \Delta(u^1) &= e^{u_{0,1}^1+u_{1,0}^1-u_{1,0}^2}, \\ \Delta(u^2) &= e^{-3u_{0,1}^1+u_{0,1}^2+u_{1,0}^2}. \end{aligned} \tag{2.16}$$

Its  $m$ -integrals:

$$\begin{aligned} F_1 &= e^{u_{1,0}^1-u_{2,0}^1} + e^{u_{-1,0}^1-u_{-2,0}^1} + e^{u_{1,0}^2+u_{1,0}^1-u_{1,0}^2-u_{2,0}^2} + e^{u_{-1,0}^1+u_{0,0}^1-u_{0,0}^2-u_{-1,0}^2} \\ &\quad + e^{u_{1,0}^2+2u_{0,0}^1-u_{0,0}^2-2u_{1,0}^1} + e^{u_{0,0}^2+2u_{0,0}^1-u_{1,0}^2-2u_{-1,0}^1} + 2e^{2u_{0,0}^1-u_{-1,0}^1-u_{1,0}^1}, \\ F_2 &= e^{u_{2,0}^2-u_{3,0}^2} + e^{3u_{0,0}^1+u_{0,0}^2-u_{1,0}^2-3u_{-1,0}^1} + 3e^{u_{0,0}^1+u_{1,0}^2-u_{2,0}^1-u_{0,0}^2} + 3e^{u_{0,0}^1+u_{1,0}^1-u_{-1,0}^1-u_{2,0}^1} \\ &\quad + 3e^{u_{1,0}^2+3u_{1,0}^1-2u_{0,0}^1-u_{2,0}^2-u_{2,0}^1} + 3e^{3u_{1,0}^1+u_{0,0}^1-2u_{2,0}^1} + 3e^{3u_{0,0}^1+u_{1,0}^1-u_{-1,0}^1} \\ &\quad + 3e^{3u_{0,0}^1+u_{1,0}^2-u_{0,0}^2-2u_{1,0}^1-u_{-1,0}^1} + 3e^{u_{1,0}^2+u_{1,0}^2-u_{2,0}^2-u_{-1,0}^1} + 2e^{2u_{1,0}^2+u_{0,0}^2-u_{2,0}^2} \\ &\quad + e^{3u_{1,0}^1+u_{2,0}^2-u_{1,0}^2-3u_{2,0}^1} + e^{3u_{1,0}^1+2u_{1,0}^2-3u_{0,0}^1-2u_{2,0}^2} + e^{3u_{0,0}^1+2u_{1,0}^2-2u_{0,0}^2-3u_{1,0}^1} + e^{u_{2,0}^2+u_{-1,0}^2}, \end{aligned} \tag{2.17}$$

### 2.3.3 System corresponding to the algebra $D_3$

The Cartan matrix of  $D_3$  is:

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The discrete system for  $D_3$  looks as:

$$\begin{aligned} \Delta(u^1) &= e^{u_{0,1}^1+u_{1,0}^1-u_{0,1}^2-u_{0,1}^3}, \\ \Delta(u^2) &= e^{u_{0,1}^2+u_{1,0}^2-u_{1,0}^1}, \\ \Delta(u^3) &= e^{u_{0,1}^3+u_{1,0}^3-u_{1,0}^1}. \end{aligned} \tag{2.18}$$

Its  $m$ -integrals:

$$\begin{aligned}
F_1 &= e^{u_{0,0}^3 - u_{-1,0}^3} + e^{u_{1,0}^2 - u_{2,0}^2} + e^{u_{1,0}^2 + u_{1,0}^1 - u_{2,0}^1 - u_{0,0}^2} + e^{u_{1,0}^1 + u_{0,0}^3 - u_{0,0}^1 - u_{1,0}^3}, \\
F_2 &= e^{u_{0,0}^2 - u_{-1,0}^2} + e^{u_{1,0}^3 - u_{2,0}^3} + e^{u_{1,0}^3 + u_{1,0}^1 - u_{2,0}^1 - u_{0,0}^3} + e^{u_{1,0}^1 + u_{0,0}^2 - u_{0,0}^1 - u_{1,0}^2}, \\
F_3 &= e^{u_{1,0}^1 - u_{2,0}^1} + e^{u_{0,0}^1 - u_{-1,0}^1} + e^{u_{0,0}^2 - u_{1,0}^2 + u_{0,0}^3 - u_{-1,0}^3} + e^{u_{0,0}^2 - u_{-1,0}^2 + u_{0,0}^3 - u_{1,0}^3} \\
&\quad + e^{u_{1,0}^1 - u_{0,0}^1 + u_{0,0}^2 - u_{1,0}^2 + u_{0,0}^3 - u_{1,0}^3} + e^{u_{0,0}^1 - u_{1,0}^1 + u_{0,0}^2 - u_{-1,0}^2 + u_{0,0}^3 - u_{-1,0}^3}.
\end{aligned} \tag{2.19}$$

To find the integrals in the other direction one uses the discrete symmetry  $n \rightarrow -m$ ,  $m \rightarrow -n$  of the system.

### 3 Symmetries of discrete systems

In this section we demonstrate that the discrete systems admit generalized symmetries. For the case of the simple Lie algebras the systems admit local symmetries while for the case of affine algebras the symmetries are nonlocal.

#### 3.1 Generalized symmetries for the system $A_2$

Higher symmetries for system corresponding to  $A_2$  are found by using the method suggested in [29], (see also [30]). The lowest order symmetry is of the form:

$$\begin{aligned}
\frac{du_{0,0}^1}{dt} &= \frac{3e^{u_{0,0}^1 - u_{1,0}^1}}{D_m^{-1}F_1} - 1, \\
\frac{du_{0,0}^2}{dt} &= 1 - \frac{3e^{u_{0,0}^2 - u_{-1,0}^2}}{D_m^{-1}F_1}.
\end{aligned}$$

The next order symmetry depends on arbitrary functions  $f(x, y)$ ,  $g(x)$  of two and respectively one variable and has the form:

$$\begin{aligned}
\frac{du_{0,0}^1}{dt} &= \frac{f(F_1, D_m^{-1}F_2) + g(D_m^{-1}F_1)}{D_m^{-1}F_2} e^{u_{0,0}^1 - u_{-1,0}^1} + f(F_1, D_m^{-1}F_2) - g(F_1) + g(D_m^{-1}F_1) - \\
&\quad \frac{f(F_1, D_m^{-1}F_2) + f(D_m^{-2}F_1, D_m^{-2}F_2) - g(F_1)}{D_m^{-1}F_1} e^{u_{0,0}^1 - u_{1,0}^1}, \\
\frac{du_{0,0}^2}{dt} &= \frac{f(F_1, D_m^{-1}F_2) + f(D_m^{-2}F_1, D_m^{-2}F_2) - g(F_1)}{D_m^{-1}F_1} e^{u_{0,0}^2 - u_{-1,0}^2} - f(D_m^{-2}F_1, D_m^{-2}F_2) + \\
&\quad \frac{f(D_m^{-1}F_1, D_m^{-2}F_2) + g(D_m^{-2}F_1)}{D_m^{-2}F_2} e^{u_{0,0}^2 - u_{1,0}^2}.
\end{aligned}$$

### 3.2 Evaluation of a generalized symmetry for the system $A_1^{(1)}$

The method for searching nonlocal symmetries is illustrated with the following example:

$$\begin{aligned} e^{-u_{1,1}-u_{0,0}} - e^{-u_{1,0}-u_{0,1}} &= e^{-2v_{1,0}}, \\ e^{-v_{1,1}-v_{0,0}} - e^{-v_{1,0}-v_{0,1}} &= e^{-2u_{0,1}}, \end{aligned} \quad (3.1)$$

corresponding to the algebra  $A_1^{(1)}$  with the Cartan matrix:

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Recall that the continuous version of the system is:

$$u_{xy} = \exp(2u - 2v), \quad v_{xy} = \exp(2v - 2u). \quad (3.2)$$

It can be proved that system (3.2) does not have any local generalized symmetry, however it has nonlocal generalized symmetries, the simplest one can be represented in terms of  $s = u + v$ ,  $r = u - v$  as follows:

$$r_t = r_{xxx} - 2r_x^3, \quad s_{xt} = 2r_x r_{xxx} - 3r_x^4 - r_{xx}^2 + F(W, W_x). \quad (3.3)$$

Here  $F$  is an arbitrary function and  $W = s_{xx} - r_x^2$  is the  $y$ -integral of system (3.2). Note that years ago in [9] it was observed that equation  $r_t = r_{xxx} - 2r_x^3$  is consistent with the system (3.2).

For the fully discrete analogue (3.1) of the system (3.2) we have a very similar situation. The system (3.1) does not have any local generalized symmetry. One has to look for a nonlocal symmetry. The main trouble arising in this case is connected with the guessing of the structure of non-locality and the form of the symmetry.

Rewrite system (3.1) in the form:

$$\begin{aligned} a_{1,1}a_{0,0} - a_{1,0}a_{0,1} &= b_{1,0}^2, \\ b_{1,1}b_{0,0} - b_{1,0}b_{0,1} &= a_{0,1}^2, \end{aligned} \quad (3.4)$$

where  $a_{i,j} = \exp(-u_{i,j})$ ,  $b_{i,j} = \exp(-v_{i,j})$ .

We will look for the generalized symmetry of (3.4) in such a nonlocal form:

$$\begin{aligned} \partial_t a_{1,0} &= f_1(a_{0,0}, a_{1,0})\partial_t a_{0,0} + f(a_{-1,0}, a_{0,0}, a_{1,0}, a_{2,0}, b_{-1,0}, b_{0,0}, b_{1,0}, b_{2,0}), \\ \partial_t b_{1,0} &= g_1(b_{0,0}, b_{1,0})\partial_t b_{0,0} + g(a_{-1,0}, a_{0,0}, a_{1,0}, a_{2,0}, b_{-1,0}, b_{0,0}, b_{1,0}, b_{2,0}). \end{aligned} \quad (3.5)$$

Here  $f_1, g_1, f, g$  are unknown functions.

From (3.5) we can find:

$$\partial_t a_{1,1} = D_m(f_1)\partial_t a_{0,1} + D_m f, \quad \partial_t b_{1,1} = D_m(g_1)\partial_t b_{0,1} + D_m g. \quad (3.6)$$

After differentiation (3.1) with respect to  $t$  by means of the systems (3.5), (3.6) we get:

$$\begin{aligned} \partial_t a_{0,0}(a_{1,1} - a_{0,1}f_1) + \partial_t a_{0,1}(a_{0,0}D_m f_1 - a_{1,0}) - 2\partial_t b_{0,0}b_{1,0}g_1 &= \dots, \\ -2a_{0,1}\partial_t a_{0,1} + \partial_t b_{0,0}(b_{1,1} - b_{0,1}g_1) + \partial_t b_{0,1}(b_{0,0}D_m g_1 - b_{1,0}) &= \dots \end{aligned} \quad (3.7)$$

Here the right hand sides do not depend on the derivatives of dynamical variables with respect to  $t$ . By applying the operator  $D_n$  to both sides of (3.7) we obtain:

$$\begin{aligned} \partial_t a_{0,0}f_1 D_n(a_{1,1} - a_{0,1}f_1) + \partial_t a_{0,1}D_m f_1 D_n(a_{0,0}D_m f_1 - a_{1,0}) - 2\partial_t b_{1,0}b_{2,0}g_1 D_n g_1 &= \dots, \\ -2a_{1,1}\partial_t D_m f_1 + \partial_t b_{0,0}g_1 D_n(b_{1,1} - b_{0,1}g_1) + \partial_t b_{0,1}D_m g_1 D_n(b_{0,0}D_m g_1 - b_{1,0}) &= \dots \end{aligned} \quad (3.8)$$

System of equations (3.7), (3.8) is a system of linear algebraic equations with unknowns  $\partial_t a_{0,0}$ ,  $\partial_t b_{0,0}$ ,  $\partial_t a_{0,1}$ ,  $\partial_t b_{0,1}$ . If the determinant of this system is different from zero, then due to the Cramer's rule the system has unique solution, and therefore the searched generalized symmetry is local. This output is in contradiction with our previous study proving the absence of local symmetries. Thus the determinant should be zero:

$$\begin{vmatrix} a_{1,1} - f_1 a_{0,1} & a_{0,0}D_m f_1 - a_{1,0} & -2b_{1,0}g_1 & 0 \\ f_1(a_{2,1} - a_{1,1}D_n f_1) & (a_{1,0}D_n D_m f_1 - a_{2,0})D_m f_1 & -2b_{2,0}g_1 D_n g_1 & 0 \\ 0 & -2a_{1,1}D_m f_1 & g_1(b_{2,1} - b_{1,1}D_n g_1) & D_m g_1(b_{1,0}D_n D_m g_1 - b_{2,0}) \\ 0 & -2a_{0,1} & b_{1,1} - g_1 b_{0,1} & b_{0,0}D_m g_1 - b_{1,0} \end{vmatrix}. \quad (3.9)$$

Vanishing of the determinant implies

$$f_1(a_{0,0}, a_{1,0}) = a_{1,0}/a_{0,0}, \quad g_1(b_{0,0}, b_{1,0}) = b_{1,0}/b_{0,0}, \quad (3.10)$$

In virtue of (3.10) equations (3.7), (3.8) take the form:

$$\begin{aligned}
\frac{\partial_t a_{0,0}}{a_{0,0}} + \frac{\partial_t a_{0,1}}{a_{0,1}} - 2 \frac{\partial_t b_{0,0}}{b_{0,0}} &= \frac{2}{b_{1,0}} g + \frac{a_{0,1}}{b_{1,0}^2} f - \frac{a_{0,0}}{b_{1,0}^2} D_m f, \\
\frac{\partial_t a_{0,0}}{a_{0,0}} + \frac{\partial_t a_{0,1}}{a_{0,1}} - 2 \frac{\partial_t b_{0,0}}{b_{0,0}} &= \frac{2}{b_{2,0}} D_n g + \frac{a_{1,1}}{b_{2,0}^2} D_n f - \frac{a_{1,0}}{b_{2,0}^2} D_n D_m f - \frac{f}{a_{1,0}} - \frac{D_m f}{a_{1,1}} + \frac{2}{b_{1,0}} g, \\
\frac{\partial_t b_{0,0}}{b_{0,0}} + \frac{\partial_t b_{0,1}}{b_{0,1}} - 2 \frac{\partial_t a_{0,1}}{a_{0,1}} &= \frac{b_{0,1}}{a_{0,1}^2} g + \frac{b_{0,0}}{a_{0,1}^2} D_m g, \\
\frac{\partial_t b_{0,0}}{b_{0,0}} + \frac{\partial_t b_{0,1}}{b_{0,1}} - 2 \frac{\partial_t a_{0,1}}{a_{0,1}} &= \frac{b_{1,1}}{a_{1,1}^2} D_n g + \frac{b_{1,0}}{a_{1,1}^2} D_n D_m g + \frac{2}{a_{1,1}} D_m f - \frac{g}{b_{1,0}} - \frac{D_m g}{b_{1,1}}.
\end{aligned} \tag{3.11}$$

We can see that the left hand sides of the first and third equations as well as the left hand sides of the second and fourth equations of the last system are identical. Comparison of their right hand sides gives a system of two equation for  $f, g$ :

$$\frac{a_{1,0}}{b_{2,0}^2} D_n D_m f - \frac{a_{1,1}}{b_{2,0}^2} D_n f - \frac{a_{1,0} a_{0,1}}{a_{1,1} b_{1,0}^2} D_m f + \frac{a_{1,1} a_{0,0}}{a_{1,0} b_{1,0}^2} f - \frac{2}{b_{2,0}} D_n g = 0, \tag{3.12}$$

$$\frac{b_{1,0}}{a_{1,1}^2} D_n D_m g + \frac{b_{1,1}}{a_{1,1}^2} D_n g - \frac{b_{1,0} b_{0,1}}{b_{1,1} a_{0,1}^2} D_m g - \frac{b_{1,1} b_{0,0}}{b_{1,0} a_{0,1}^2} g + \frac{2}{a_{1,1}} D_m f = 0. \tag{3.13}$$

Differentiate (3.12) and (3.13) with respect to  $a_{3,0}$ ,  $b_{3,0}$ ,  $b_{-1,0}$  or  $a_{-1,0}$  and get:

$$\frac{b_{0,0} a_{0,0}}{a_{1,0}^2} \frac{\partial f}{\partial b_{-1,0}} = D_m \left( \frac{b_{0,0} a_{0,0}}{a_{1,0}^2} \frac{\partial f}{\partial b_{-1,0}} \right), \quad D_m \left( \frac{a_{0,0}^2}{a_{1,0}^2} \frac{\partial f}{\partial a_{-1,0}} \right) = \frac{a_{0,0}^2}{a_{1,0}^2} \frac{\partial f}{\partial a_{-1,0}} + \frac{2 a_{-1,1} a_{0,1}^2}{a_{1,1}^2 b_{0,0}} D_m \frac{\partial f}{\partial b_{-1,0}}, \tag{3.14}$$

$$\begin{aligned}
\frac{a_{1,0}}{b_{0,0}} \frac{\partial g}{\partial a_{2,0}} &= \frac{a_{1,1}}{b_{0,1}} D_m \frac{\partial g}{\partial a_{2,0}}, \quad D_m \frac{\partial g}{\partial b_{2,0}} + \frac{2 b_{1,0} b_{2,0}}{a_{1,0} b_{1,1}} D_m \frac{\partial g}{\partial a_{2,0}} = \frac{b_{1,0} b_{0,1}}{b_{0,0} b_{1,1}} \frac{\partial g}{\partial b_{2,0}}, \\
\frac{\partial f}{\partial a_{2,0}} + \frac{2 b_{1,0}}{a_{0,1}} \frac{\partial g}{\partial a_{2,0}} &= \frac{a_{1,1} a_{0,0}}{a_{1,0} a_{0,1}} D_m \frac{\partial f}{\partial a_{2,0}}, \quad \frac{\partial f}{\partial b_{2,0}} + \frac{2 b_{1,0}}{a_{0,1}} \frac{\partial g}{\partial b_{2,0}} = \frac{2 a_{0,0} b_{2,0}}{a_{1,0} a_{0,1}} D_m \frac{\partial f}{\partial a_{2,0}} + \frac{a_{0,0} b_{1,1}}{a_{0,1} b_{1,0}} D_m \frac{\partial f}{\partial b_{2,0}},
\end{aligned} \tag{3.15}$$

$$\frac{\partial g}{\partial b_{-1,0}} + \frac{2 b_{1,0} a_{0,0}^2}{a_{1,0} b_{1,-1} b_{0,0}} \frac{\partial f}{\partial b_{-1,0}} = \frac{b_{1,0}^2 b_{0,-1}^2}{b_{1,-1}^2 b_{0,0}^2} D_m^{-1} \frac{\partial g}{\partial b_{-1,0}}, \tag{3.17}$$

$$\frac{\partial g}{\partial a_{-1,0}} + \frac{2 a_{0,0}^2 b_{1,0}}{a_{1,0} b_{1,-1} b_{0,0}} \frac{\partial f}{\partial a_{-1,0}} = \frac{b_{1,0}^2 a_{0,-1} b_{0,-1}}{a_{0,0} b_{0,0} b_{1,-1}} D_m^{-1} \frac{\partial g}{\partial a_{-1,0}} + \frac{4 a_{0,0}^2 a_{-1,0} b_{1,0}}{a_{1,0} b_{0,-1} b_{1,-1} b_{0,0}} \frac{\partial f}{\partial b_{-1,0}} + \frac{2 a_{-1,0}}{b_{0,-1}} \frac{\partial g}{\partial b_{-1,0}}. \tag{3.18}$$

We use the last system of eight equations in order to specify dependence of the functions  $f, g$  on the variables  $a_{-1,0}, a_{2,0}, b_{-1,0}, b_{2,0}$  corresponding to the lowest and highest values of the first index.

From the first equation in (3.14) we get:

$$\frac{\partial f}{\partial b_{-1,0}} = \frac{C_7 a_{1,0}^2}{b_{0,0} a_{0,0}}. \tag{3.19}$$



Then the second equation takes the form:

$$D_m \left( \frac{a_{0,0}^2}{a_{1,0}^2} \frac{\partial f}{\partial a_{-1,0}} \right) = \frac{a_{0,0}^2}{a_{1,0}^2} \frac{\partial f}{\partial a_{-1,0}} + \frac{2C_7 a_{-1,1} a_{0,1}}{b_{0,1} b_{0,0}}. \quad (3.20)$$

It can easily be proved that the expression  $\frac{2a_{-1,1}a_{0,1}}{b_{0,1}b_{0,0}}$  does not belong to the image of the operator  $D_m - 1$ , so we have to put  $C_7 = 0$  and then

$$\frac{\partial f}{\partial a_{-1,0}} = \frac{C_4 a_{1,0}^2}{a_{0,0}^2}. \quad (3.21)$$

From the pair of equations (3.15) we have:

$$\frac{\partial g}{\partial a_{2,0}} = 0, \quad \frac{\partial g}{\partial b_{2,0}} = \frac{C_2 b_{0,0}}{b_{1,0}}. \quad (3.22)$$

Continuing this way one can find dependence of the functions  $f, g$  on the variables  $a_{-1,0}, a_{2,0}, b_{-1,0}, b_{2,0}$ :

$$\begin{aligned} f &= \frac{a_{0,0} b_{2,0}}{b_{1,0}} C_1 + C_2 \left( \frac{a_{0,0} a_{2,0}}{a_{1,0}} + \frac{a_{0,0} b_{2,0}^2}{a_{1,0} b_{1,0}^2} + \frac{2b_{0,0} a_{1,0} b_{2,0}}{b_{1,0}^2} \right) + \frac{a_{1,0}^2 a_{-1,0}}{a_{0,0}^2} C_4 + f_1(a_{0,0}, a_{1,0}, b_{0,0}, b_{1,0}), \\ g &= \frac{b_{0,0} b_{2,0}}{b_{1,0}} C_2 + C_4 \left( \frac{b_{1,0}^2 b_{-1,0}}{b_{0,0}^2} + \frac{b_{1,0}^3 a_{-1,0}^2}{a_{0,0}^2 b_{0,0}^2} + \frac{2b_{1,0} a_{1,0} a_{-1,0}}{a_{0,0}^2} \right) + \frac{b_{1,0}^2 a_{-1,0}}{a_{0,0} b_{0,0}} C_5 + g_1(a_{0,0}, a_{1,0}, b_{0,0}, b_{1,0}). \end{aligned} \quad (3.23)$$

For the functions  $f_1, g_1$  from the same system (3.12), (3.13) we obtain the following non-homogeneous equations:

$$\frac{a_{1,0}}{b_{2,0}^2} D_n D_m f_1 - \frac{a_{1,1}}{b_{2,0}^2} D_n f_1 - \frac{a_{1,0} a_{0,1}}{a_{1,1} b_{2,0}^2} D_m f_1 + \frac{a_{1,1} a_{0,0}}{a_{1,0} b_{1,0}^2} f_1 - \frac{2}{b_{2,0}} D_n g_1 = \dots, \quad (3.24)$$

$$\frac{b_{1,0}}{a_{1,1}^2} D_n D_m g_1 + \frac{b_{1,1}}{a_{1,1}^2} D_n g_1 - \frac{b_{1,0} b_{0,1}}{b_{1,1} a_{0,1}^2} D_m g_1 - \frac{b_{1,1} b_{0,0}}{b_{1,0} a_{0,1}^2} g_1 + \frac{2}{a_{1,1}} D_m f_1 = \dots \quad (3.25)$$

Differentiating (3.25) with respect to the variable  $a_{2,0}$  we get:

$$\frac{a_{0,0}}{b_{0,0}} \frac{\partial g_1}{\partial a_{1,0}} - D_m \left( \frac{a_{0,0}}{b_{0,0}} \frac{\partial g_1}{\partial a_{1,0}} \right) = \frac{2C_2(a_{0,1}^3 a_{1,0} - b_{1,0}^3 b_{0,1})}{a_{0,0} a_{0,1} b_{1,1} b_{1,0}} + 2(C_4 - C_2) \frac{a_{0,1}^3 a_{-1,0} - b_{1,1} b_{0,0}^3}{a_{0,0} b_{0,0} a_{0,1} b_{0,1}}.$$

From this equation we can find:

$$C_4 = C_2, \quad \frac{a_{0,0}}{b_{0,0}} \frac{\partial g_1}{\partial a_{1,0}} = \frac{2C_2 a_{1,0} b_{0,0}}{a_{0,0} b_{1,0}} + C_6,$$

therefore

$$g_1 = C_2 \frac{a_{1,0}^2 b_{0,0}^2}{a_{0,0}^2 b_{1,0}} + C_6 \frac{a_{1,0} b_{0,0}}{a_{0,0}} + g_2(a_{0,0}, b_{0,0}, b_{1,0}).$$

Differentiating (3.25) with respect to  $b_{2,0}$  we get:

$$D_m \left( \frac{\partial g_2}{\partial b_{1,0}} \right) - \frac{\partial g_2}{\partial b_{1,0}} = 2(C_1 - C_6) \frac{b_{0,0}b_{1,0}}{a_{0,0}a_{0,1}} + 2(C_1 - C_5) \frac{a_{0,1}^2 a_{0,0}a_{-1,1} - b_{0,0}^2 b_{1,0}b_{0,1}}{a_{0,0}b_{0,0}a_{0,1}b_{0,1}}, \quad (3.26)$$

therefore

$$C_6 = C_1, \quad C_5 = C_1 \quad g_2 = C_3 b_{1,0} + g_3(a_{0,0}, b_{0,0}).$$

Now apply the operator  $D_n^{-1}$  to the equation (3.24) and then differentiate the obtained result with respect to the variable  $b_{-1,0}$ :

$$D_m \left( \frac{a_{0,0}b_{1,0}}{a_{1,0}^2} \frac{\partial f_1}{\partial b_{0,0}} \right) - \frac{a_{0,0}b_{1,0}}{a_{1,0}^2} \frac{\partial f_1}{\partial b_{0,0}} = 2C_2 \frac{b_{1,0}^3 b_{0,1} - a_{1,0}a_{0,1}^3}{a_{0,0}a_{0,1}b_{1,0}b_{1,1}}, \quad (3.27)$$

therefore

$$f_1 = C_2 \frac{a_{1,0}^3 b_{0,0}^2}{a_{0,0}^2 b_{1,0}^2} + C_7 \frac{a_{1,0}^2 b_{0,0}}{a_{0,0} b_{1,0}} + f_2(a_{0,0}, a_{1,0}, b_{1,0}). \quad (3.28)$$

Substitute in (3.24) the expression found instead of  $f_1$  and differentiate with respect to  $a_{2,0}$  to get:

$$D_m \left( \frac{\partial f_2}{\partial a_{1,0}} \right) - \frac{\partial f_2}{\partial a_{1,0}} = 2(C_1 - C_7) \frac{a_{1,0}a_{0,1}^3 - b_{1,0}^3 b_{0,1}}{a_{0,0}a_{0,1}b_{1,0}b_{1,1}}, \quad (3.29)$$

therefore:

$$C_7 = C_1, \quad f_2 = C_8 a_{1,0} + f_3(a_{0,0}, b_{1,0}).$$

For the functions  $f_3, g_3$  we have a system:

$$\begin{aligned} \frac{a_{1,0}}{b_{2,0}^2} D_n D_m f_3 - \frac{a_{1,1}}{b_{2,0}^2} D_n f_3 - \frac{a_{1,0}a_{0,1}}{a_{1,1}b_{2,0}^2} D_m f_3 + \frac{a_{1,1}a_{0,0}}{a_{1,0}b_{1,0}^2} f_3 - \frac{2}{b_{2,0}} D_n g_3 &= 2(C_8 - C_3), \\ \frac{b_{1,0}}{a_{1,1}^2} D_n D_m g_3 + \frac{b_{1,1}}{a_{1,1}^2} D_n g_3 - \frac{b_{1,0}b_{0,1}}{b_{1,1}a_{0,1}^2} D_m g_3 - \frac{b_{1,1}b_{0,0}}{b_{1,0}a_{0,1}^2} g_3 + \frac{2}{a_{1,1}} D_m f_3 &= 2(C_8 - C_3). \end{aligned} \quad (3.30)$$

We see that (3.30) is satisfied identically if one chooses

$$C_8 = C_3, \quad f_3 = 0, \quad g_3 = 0.$$

Thus we find the final form of the symmetry searched:

$$\begin{aligned}
\frac{\partial_t a_{1,0}}{a_{1,0}} - \frac{\partial_t a_{0,0}}{a_{0,0}} &= C_1 \left( \frac{a_{0,0} b_{2,0}}{a_{1,0} b_{1,0}} + \frac{a_{1,0} b_{0,0}}{a_{0,0} b_{1,0}} \right) \\
&\quad + C_2 \left( \frac{a_{1,0} a_{-1,0}}{a_{0,0}^2} + \frac{2b_{0,0} b_{2,0}}{b_{1,0}^2} + \frac{a_{1,0}^2 b_{0,0}^2}{a_{0,0}^2 b_{1,0}^2} + \frac{a_{0,0} a_{2,0}}{a_{1,0}^2} + \frac{a_{0,0}^2 b_{2,0}^2}{a_{1,0}^2 b_{1,0}^2} \right) + C_3, \\
\frac{\partial_t b_{1,0}}{b_{1,0}} - \frac{\partial_t b_{0,0}}{b_{0,0}} &= C_1 \left( \frac{a_{-1,0} b_{1,0}}{a_{0,0} b_{0,0}} + \frac{a_{1,0} b_{0,0}}{a_{0,0} b_{1,0}} \right) \\
&\quad + C_2 \left( \frac{2a_{1,0} a_{-1,0}}{a_{0,0}^2} + \frac{b_{0,0} b_{2,0}}{b_{1,0}^2} + \frac{a_{1,0}^2 b_{0,0}^2}{a_{0,0}^2 b_{1,0}^2} + \frac{b_{-1,0} b_{1,0}}{b_{0,0}^2} + \frac{a_{-1,0} b_{1,0}^2}{a_{0,0}^2 b_{0,0}^2} \right) + C_3.
\end{aligned} \tag{3.31}$$

Here  $C_1, C_2, C_3$  are arbitrary constants.

By analogy with the continuous case we have a differential constraint completely consistent with system (3.4), which is obtained from the last system by applying the operator  $(D_n - 1)^{-1}$  to the difference of the two equations:

$$\frac{\partial_t b_{0,0}}{b_{0,0}} - \frac{\partial_t a_{0,0}}{a_{0,0}} = C_1 \frac{a_{-1,0} b_{1,0}}{a_{0,0} b_{0,0}} + C_2 \left( \frac{a_{-1,0} a_{1,0}}{a_{0,0}^2} + \frac{b_{-1,0} b_{1,0}}{b_{0,0}^2} + \frac{a_{-1,0}^2 b_{1,0}^2}{a_{0,0}^2 b_{0,0}^2} \right) \tag{3.32}$$

Under the Cole-Hopf type transformation

$$\hat{a}_{0,0} = \frac{a_{1,0}}{a_{0,0}}, \quad \hat{b}_{0,0} = \frac{b_{1,0}}{b_{0,0}} \tag{3.33}$$

system (3.4) converts to the following one:

$$\begin{aligned}
\hat{a}_{1,1} \hat{a}_{0,0} \hat{a}_{0,1} - \hat{a}_{1,0} \hat{a}_{0,1} \hat{a}_{0,0} &= \hat{b}_{1,0}^2 (\hat{a}_{0,0} - \hat{a}_{0,1}), \\
\hat{b}_{1,1} \hat{b}_{0,0} \hat{b}_{0,1} - \hat{b}_{1,0} \hat{b}_{0,1} \hat{b}_{0,0} &= \hat{a}_{0,1}^2 (\hat{b}_{0,1} - \hat{b}_{0,0}).
\end{aligned} \tag{3.34}$$

This transformation brings our symmetry to a local symmetry of the new system (3.34):

$$\begin{aligned}
\partial_t \hat{a}_{0,0} &= C_1 \left( \hat{b}_{1,0} + \frac{\hat{a}_{0,0}^2}{\hat{b}_{0,0}} \right) + C_2 \left( \frac{\hat{a}_{0,0}^2}{\hat{a}_{-1,0}} + \frac{2\hat{b}_{1,0} \hat{a}_{0,0}}{\hat{b}_{0,0}} + \frac{\hat{a}_{0,0}^3}{\hat{b}_{0,0}^2} + \hat{a}_{1,0} + \frac{\hat{b}_{1,0}^2}{\hat{a}_{0,0}} \right) + C_3 \hat{a}_{0,0}, \\
\partial_t \hat{b}_{0,0} &= C_1 \left( \frac{\hat{b}_{0,0}^2}{\hat{a}_{-1,0}} + \hat{a}_{0,0} \right) + C_2 \left( \frac{2\hat{a}_{0,0} \hat{b}_{0,0}}{\hat{a}_{-1,0}} + \hat{b}_{1,0} + \frac{\hat{a}_{0,0}^2}{\hat{b}_{0,0}} + \frac{\hat{b}_{0,0}^2}{\hat{b}_{-1,0}} + \frac{\hat{b}_{0,0}^3}{\hat{a}_{-1,0}^2} \right) + C_3 \hat{b}_{0,0}
\end{aligned} \tag{3.35}$$

### 3.3 Generalized symmetries for the system $A_2^{(2)}$

The Cartan matrix of  $A_2^{(2)}$  is:

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

The discrete system for  $A_2^{(2)}$  looks as:

$$\begin{aligned}\Delta(u^1) &= e^{u_{0,1}^1 + u_{1,0}^1 - u_{1,0}^2}, \\ \Delta(u^2) &= e^{-4u_{0,1}^1 + u_{0,1}^2 + u_{1,0}^2}.\end{aligned}\tag{3.36}$$

Generalized symmetry

$$\begin{aligned}(u_{1,0}^1 - u_{0,0}^1)_t &= C_1 \left( e^{2u_{1,0}^1 - u_{0,0}^1 - u_{2,0}^1} + e^{2u_{1,0}^1 - 2u_{0,0}^1 + u_{1,0}^2 - u_{2,0}^2} + e^{2u_{0,0}^1 - 2u_{1,0}^1 + u_{1,0}^2 - u_{0,0}^2} + e^{2u_{0,0}^1 - u_{1,0}^1 - u_{-1,0}^1} \right) \\ &+ C_2 \left( 3e^{4u_{1,0}^1 - 2u_{0,0}^1 - 2u_{2,0}^1} + e^{u_{1,0}^1 - u_{0,0}^1 + u_{2,0}^1 - u_{3,0}^1} + 3e^{-u_{2,0}^1 + u_{1,0}^2 + u_{0,0}^1 - u_{0,0}^2} \right. \\ &+ 3e^{4u_{1,0}^1 - u_{2,0}^1 - 3u_{0,0}^1 + u_{1,0}^2 - u_{2,0}^2} + 2e^{u_{1,0}^1 - u_{2,0}^1 + u_{0,0}^1 - u_{-1,0}^1} + e^{4u_{1,0}^1 - 4u_{0,0}^1 + 2u_{1,0}^2 - 2u_{2,0}^2} \\ &+ 2e^{u_{2,0}^2 - 2u_{1,0}^2 - u_{0,0}^2} + 3e^{u_{1,0}^1 - u_{2,0}^2 + u_{1,0}^2 - u_{-1,0}^1} + e^{-u_{1,0}^1 + u_{-1,0}^1 + u_{0,0}^1 - u_{-2,0}^1} \\ &+ e^{-4u_{1,0}^1 + 2u_{1,0}^2 - 2u_{0,0}^2 + 4u_{0,0}^1} + e^{4u_{1,0}^1 - u_{0,0}^1 + u_{2,0}^2 - 3u_{2,0}^1 - u_{1,0}^2} + e^{-u_{1,0}^1 + u_{-1,0}^1 + u_{0,0}^2 - u_{-1,0}^2} \\ &+ 3e^{-2u_{1,0}^1 - 2u_{-1,0}^1 + 4u_{0,0}^1} + e^{-u_{0,0}^1 + u_{2,0}^2 + u_{2,0}^1 - u_{3,0}^2} + 3e^{3u_{1,0}^1 + u_{1,0}^2 - u_{-1,0}^1 - u_{0,0}^2 + 4u_{0,0}^1} \\ &\left. + e^{-u_{1,0}^1 - u_{1,0}^2 - 3u_{-1,0}^1 + u_{0,0}^2 + 4u_{0,0}^1} \right) + C_3, \\ (u_{1,0}^2 - u_{0,0}^2)_t &= 2C_1 \left( e^{-2u_{1,0}^1 + u_{1,0}^2 - u_{0,0}^2 + 2u_{0,0}^1} + 2e^{-u_{1,0}^1 - u_{-1,0}^1 + 2u_{0,0}^1} + e^{-u_{1,0}^2 + 2u_{0,0}^1 + u_{0,0}^2 - 2u_{-1,0}^1} \right) \\ &+ 2C_2 \left( 2e^{u_{1,0}^1 - u_{2,0}^1 + u_{0,0}^1 - u_{-1,0}^1} + 2e^{-u_{2,0}^1 + u_{1,0}^2 + u_{0,0}^1 - u_{0,0}^2} + 4e^{-3u_{1,0}^1 + u_{1,0}^2 - u_{-1,0}^1 - u_{0,0}^2 + 4u_{0,0}^1} \right. \\ &+ e^{-4u_{1,0}^1 + 2u_{1,0}^2 - 2u_{0,0}^2 + 4u_{0,0}^1} + 2e^{u_{1,0}^1 - u_{2,0}^2 + u_{1,0}^2 - u_{-1,0}^1} + e^{-u_{2,0}^2 + 2u_{1,0}^2 - u_{0,0}^2} \\ &+ 2e^{-u_{1,0}^1 + u_{-1,0}^1 + u_{0,0}^1 - u_{-2,0}^1} + 2e^{-u_{1,0}^1 + u_{-1,0}^1 + u_{0,0}^2 - u_{-2,0}^1} + 4e^{-u_{1,0}^1 - u_{1,0}^2 - 3u_{-1,0}^1 + u_{0,0}^2 + 4u_{0,0}^1} \\ &+ 2e^{-u_{1,0}^2 + u_{0,0}^1 + u_{0,0}^2 - u_{-2,0}^1} + e^{-u_{1,0}^2 + 2u_{0,0}^2 - u_{-1,0}^2} + 6e^{-2u_{1,0}^1 - 2u_{-1,0}^1 + 4u_{0,0}^1} \\ &\left. + e^{-2u_{1,0}^2 + 4u_{0,0}^1 + 2u_{0,0}^2 - 4u_{-1,0}^1} \right) + 2C_3.\end{aligned}\tag{3.37}$$

Here  $C_1, C_2, C_3$  are arbitrary constants. And for combination we have a local constraint:

$$\begin{aligned}(2u_{0,0}^1 - u_{0,0}^2)_t &= 2C_1 \left( e^{2u_{0,0}^1 - u_{1,0}^1 - u_{-1,0}^1} + e^{2u_{0,0}^1 - 2u_{-1,0}^1 + u_{0,0}^2 - u_{1,0}^2} \right) + \\ &2C_2 \left( e^{-u_{-1,0}^1 - u_{2,0}^1 + u_{0,0}^1 + u_{1,0}^1} + e^{u_{1,0}^1 + u_{1,0}^2 - u_{-1,0}^1 - u_{2,0}^2} + e^{-3u_{1,0}^1 + u_{1,0}^2 + 4u_{0,0}^1 - u_{-1,0}^1 - u_{0,0}^2} \right. \\ &+ 3e^{-2u_{1,0}^1 + 4u_{0,0}^1 - 2u_{-1,0}^1} + 3e^{-u_{1,0}^1 - u_{1,0}^2 + 4u_{0,0}^1 - 3u_{-1,0}^1 + u_{0,0}^2} + e^{-u_{1,0}^1 + u_{-1,0}^1 + u_{0,0}^2 - u_{-2,0}^1} \\ &+ e^{-u_{1,0}^1 + u_{0,0}^1 + u_{-1,0}^1 - u_{-2,0}^1} + e^{-2u_{1,0}^2 + 4u_{0,0}^1 - 4u_{-1,0}^1 + 2u_{0,0}^2} + e^{-u_{1,0}^2 + 2u_{0,0}^2 - u_{-2,0}^1} \\ &\left. + 2e^{-u_{1,0}^2 + u_{0,0}^1 + u_{0,0}^2 - u_{-2,0}^1} \right).\end{aligned}\tag{3.38}$$

## 4 Characteristic $m$ - algebra for the case $A_2$

Let us describe briefly the properties of the characteristic Lie algebras of the system

$$\begin{aligned} a_{0,0}a_{1,1} &= a_{1,0}a_{0,1} + b_{1,0}, \\ b_{0,0}b_{1,1} &= b_{1,0}b_{0,1} + a_{0,1}. \end{aligned} \quad (4.1)$$

System (4.1) corresponds to the simple Lie algebra  $A_2$ . First we introduce the notion of the characteristic  $m$ -algebra for the system (4.1). Recall that according the definition an  $m$ -integral  $F(a_{0,0}, b_{0,0}, a_{1,0}, b_{1,0}, a_{-1,0}, b_{-1,0}, \dots)$  should satisfy the equation  $D_m F = F$ . In the coordinate representation this condition reads

$$F(a_{0,1}, b_{0,1}, a_{1,1}, b_{1,1}, a_{-1,1}, b_{-1,1}, \dots) = F(a_{0,0}, b_{0,0}, a_{1,0}, b_{1,0}, a_{-1,0}, b_{-1,0}, \dots). \quad (4.2)$$

Evidently the right hand side in (4.2) does not depend on the variables  $a_{0,1}$  and  $b_{0,1}$  hence the conditions hold  $\frac{\partial}{\partial a_{0,1}} D_m^{-1} F = 0$ ,  $\frac{\partial}{\partial b_{0,1}} D_m^{-1} F = 0$  as well as  $Y_1 F = 0$ ,  $Y_2 F = 0$  where  $Y_1 := D_m^{-1} \frac{\partial}{\partial a_{0,1}} D_m$ ,  $Y_2 := D_m^{-1} \frac{\partial}{\partial b_{0,1}} D_m$ . Denote through  $L_n$  the Lie algebra generated by the operators  $X_1 = \frac{\partial}{\partial a_{0,-1}}$ ,  $X_2 = \frac{\partial}{\partial b_{0,-1}}$ ,  $Y_1$ ,  $Y_2$ . Here the first two operators are the first order linear differential operators, or vector fields. The last two operators can also be rewritten as vector fields of the form

$$\begin{aligned} Y_1 = \frac{\partial}{\partial a_{0,0}} + \left( \frac{a_{1,0}}{a_{0,0}} - \frac{b_{1,0}b_{0,-1}}{a_{0,0}b_{0,0}a_{0,-1}} \right) \frac{\partial}{\partial a_{1,0}} + \left( \frac{a_{-1,0}}{a_{0,0}} + \frac{b_{0,-1}}{a_{0,0}a_{0,-1}} \right) \frac{\partial}{\partial a_{-1,0}} + \dots \\ + \frac{1}{b_{0,-1}} \frac{\partial}{\partial b_{1,0}} - \left( \frac{a_{-1,0}}{a_{0,0}b_{0,-1}} + \frac{1}{a_{0,0}a_{0,-1}} \right) \frac{\partial}{\partial b_{-1,0}} + \dots, \end{aligned} \quad (4.3)$$

$$Y_2 = \frac{\partial}{\partial b_{0,0}} + \left( \frac{b_{1,0}}{b_{0,0}} - \frac{a_{0,0}}{b_{0,0}b_{0,-1}} \right) \frac{\partial}{\partial b_{1,0}} + \left( \frac{b_{-1,0}}{b_{0,0}} + \frac{a_{-1,0}}{b_{0,0}b_{0,-1}} \right) \frac{\partial}{\partial b_{-1,0}} + \dots \quad (4.4)$$

Taking commutators of the vector fields  $X_1, X_2, Y_1, Y_2$  we get vector fields

$$P_1 = [X_1, Y_1], P_2 = [X_2, Y_1], P_3 = [X_2, Y_2], P_4 = [X_1, Y_2].$$

Obviously the vector fields  $X_1, X_2, Y_1, Y_2, P_1, P_2, P_3, P_4$  belong to the  $m$ -algebra. We use the following lemma to show that  $m$ -algebra is of finite dimension.

**Lemma 4.1** Suppose that the vector field

$$K = \sum_{k=1}^{\infty} \left( \alpha_k \frac{\partial}{\partial a_k} + \alpha_{-k} \frac{\partial}{\partial a_{-k}} \right) + \sum_{k=1}^{\infty} \left( \beta_k \frac{\partial}{\partial b_k} + \beta_{-k} \frac{\partial}{\partial b_{-k}} \right) \quad (4.5)$$

satisfies the equality  $D_n K D_n^{-1} = hK$ , where  $h$  is a function depending on shifts and derivatives of variables  $a$  and  $b$ , then  $K = 0$ .

One can easily check that

$$\begin{aligned} D_n X_1 D_n^{-1} &= \frac{a_{0,0}}{a_{1,0}} X_1, & D_n X_2 D_n^{-1} &= \frac{1}{a_{1,0}} X_1 + \frac{b_{0,0}}{b_{1,0}} X_2, \\ D_n Y_1 D_n^{-1} &= \frac{a_{0,-1}}{a_{1,-1}} Y_1 - \frac{a_{0,-1}}{a_{1,-1} b_{1,-1}} Y_2, & D_n Y_2 D_n^{-1} &= \frac{b_{0,-1}}{b_{1,-1}} Y_2. \end{aligned}$$

Put  $\tilde{X}_1 = a_{0,0} X_1$ ,  $\tilde{X}_2 = b_{0,0} X_2$ ,  $\tilde{Y}_1 = a_{0,-1} Y_1$ ,  $\tilde{Y}_2 = b_{0,-1} Y_2$ , then

$$\begin{aligned} D_n \tilde{X}_1 D_n^{-1} &= \tilde{X}_1, & D_n \tilde{X}_2 D_n^{-1} &= \frac{b_{1,0}}{a_{0,0} a_{1,0}} \tilde{X}_1 + \tilde{X}_2, \\ D_n \tilde{Y}_1 D_n^{-1} &= \tilde{Y}_1 - \frac{a_{0,-1}}{b_{0,-1} b_{1,-1}} \tilde{Y}_2, & D_n \tilde{Y}_2 D_n^{-1} &= \tilde{Y}_2. \end{aligned}$$

**Lemma 4.2**

$$\begin{aligned} D_n P_1 D_n^{-1} &= P_1 - \frac{a_{0,0}}{b_{0,-1} b_{1,-1}} \tilde{Y}_2, \\ D_n P_2 D_n^{-1} &= P_2 + \frac{b_{1,0}}{a_{0,0} a_{1,0}} P_1 - \frac{a_{0,-1}}{b_{0,-1} b_{1,-1}} P_3 + \\ &\quad + \left( \frac{a_{0,-1} b_{1,0}}{a_{0,0}^2 a_{1,0}} + \frac{a_{1,-1} b_{1,0}}{a_{0,0} a_{1,0}^2} \right) \tilde{X}_1 + \left( \frac{a_{0,-1} b_{0,0}}{b_{0,-1}^2 b_{1,-1}} + \frac{a_{1,-1} a_{0,0} b_{1,0}}{a_{1,0} b_{0,-1} b_{1,-1}^2} \right) \tilde{Y}_2, \\ D_n P_3 D_n^{-1} &= P_3 - \frac{b_{1,-1}}{a_{0,0} a_{1,0}} \tilde{X}_1. \end{aligned} \quad (4.6)$$

**Lemma 4.3** For  $c = 1, 2, 3$  we have  $P_4 = 0$ .

	$\tilde{X}_1$	$\tilde{X}_2$	$\tilde{Y}_1$	$\tilde{Y}_2$	$P_1$	$P_2$	$P_3$
$\tilde{X}_1$	0	0	$P_1$	0	0	$R_1$	0
$\tilde{X}_2$	0	0	$P_2$	$P_3$	$R_1$	$R_2$	$-2\tilde{X}_2$
$\tilde{Y}_1$	$-P_1$	$-P_2$	0	0	$2\tilde{Y}_1$	$R_3$	$R_4$
$\tilde{Y}_2$	0	$-P_3$	0	0	0	$R_4$	0
$P_1$	0	$-R_1$	$-2\tilde{Y}_1$	0	0	$Q_1$	$Q_2$
$P_2$	$-R_1$	$-R_2$	$-R_3$	$-R_4$	$-Q_1$	0	$Q_3$
$P_3$	0	$2\tilde{X}_2$	$-R_4$	0	$-Q_2$	$-Q_3$	0

$$\begin{aligned}
R_1 &= \tilde{X}_2 - \frac{b_{1,0}}{b_{0,-1}} P_1 + \frac{b_{0,0}^2}{b_{0,-1}^2} \tilde{Y}_2, \\
R_2 &= \frac{2a_{0,-1}b_{0,0}^2}{a_{0,0}b_{0,-1}^2} P_3 - \frac{2a_{0,-1}b_{0,0}^3}{a_{0,0}b_{0,-1}^3} \tilde{Y}_2 + \frac{2a_{0,-1}b_{0,0}}{a_{0,0}b_{0,-1}} \tilde{X}_2, \\
R_3 &= \frac{2a_{0,-1}^2b_{0,0}}{a_{0,0}^2b_{0,-1}} P_1 - \frac{2a_{0,-1}b_{0,0}}{a_{0,0}b_{0,-1}} \tilde{Y}_1 + \frac{2a_{0,-1}^3b_{0,0}}{a_{0,0}^3b_{0,-1}} \tilde{X}_1, \\
R_4 &= -\tilde{Y}_1 - \frac{a_{0,-1}^2}{a_{0,0}^2} \tilde{X}_1 - \frac{a_{0,-1}}{a_{0,0}} P_1, \\
Q_1 &= \frac{a_{0,-1}b_{0,0}}{a_{0,0}b_{0,-1}} P_1 - 3P_2 - P_3 + \frac{a_{0,-1}^2b_{0,0}}{a_{0,0}^2b_{0,-1}} \tilde{X}_1 + \frac{b_{0,0}}{b_{0,-1}} \tilde{Y}_1, \\
Q_2 &= \frac{2b_{0,0}}{b_{0,-1}} \tilde{Y}_2 - 2P_3, \\
Q_3 &= -3P_2 + \frac{a_{0,-1}b_{0,0}}{a_{0,0}b_{0,-1}} P_3 - \frac{a_{0,-1}}{a_{0,0}} \tilde{X}_2 - \frac{a_{0,-1}b_{0,0}^2}{a_{0,0}b_{0,-1}^2} \tilde{Y}_2.
\end{aligned}$$

Any  $m$ -integral  $F(\dots a_{0,0}, b_{0,0}, a_{1,0}, b_{1,0}, a_{2,0}, b_{2,0} \dots)$  must satisfy equations

$$\begin{aligned}
\tilde{X}_1(F) &= 0, \quad \tilde{X}_2(F) = 0, \quad \tilde{Y}_1(F) = 0, \quad \tilde{Y}_2(F) = 0, \\
P_1(F) &= 0, \quad P_2(F) = 0, \quad P_3(F) = 0.
\end{aligned} \tag{4.7}$$

Solving system (4.7), it is enough to assume that  $F_1$  depends on  $b_{0,0}, a_{0,0}, b_{1,0}, a_{1,0}, b_{2,0}, a_{2,0}$  and  $F_2$  depends on  $a_{-1,0}, b_{0,0}, a_{0,0}, b_{1,0}, a_{1,0}, b_{2,0}$ , we find two independent  $m$ -integrals

$$F_1 = \frac{b_{0,0}}{b_{1,0}} + \frac{a_{0,0}b_{2,0}}{a_{1,0}b_{1,0}} + \frac{a_{2,0}}{a_{1,0}}, \quad F_2 = \frac{a_{-1,0}}{a_{0,0}} + \frac{a_{1,0}b_{0,0}}{a_{0,0}b_{1,0}} + \frac{b_{2,0}}{b_{1,0}}. \tag{4.8}$$

## 5 Cutting off constraints for the Hirota equation and discrete Zakharov-Shabat systems

It is well known that Hirota chain

$$t_{0,0}^j t_{1,1}^j - t_{1,0}^j t_{0,1}^j = t_{1,0}^{j-1} t_{0,1}^{j+1} \tag{5.1}$$

admits the Lax pair consisting of two linear discrete equations

$$\psi_{1,0}^j = \frac{t_{1,0}^{j+1} t_{0,0}^j}{t_{0,0}^{j+1} t_{1,0}^j} \psi_{0,0}^j - \psi_{0,0}^{j+1}, \quad \psi_{0,1}^j = \psi_{0,0}^j + \frac{t_{0,1}^{j+1} t_{0,0}^{j-1}}{t_{0,0}^j t_{0,1}^j} \psi_{0,0}^{j-1}. \tag{5.2}$$

Here the lower indices mean as previously shifts of the arguments, and the upper index enumerates the field variables  $t^j$  and eigenfunctions  $\psi^j$ . Exclude from the system of equations (5.2) all the eigenfunctions with the upper index different from  $j$ . As a result one gets a linear discrete

hyperbolic equation for  $\psi^j$

$$\psi_{1,1}^j - \psi_{1,0}^j - \frac{t_{1,1}^{j+1}t_{0,1}^j}{t_{0,1}^{j+1}t_{1,1}^j}\psi_{0,1}^j + \frac{t_{0,0}^jt_{1,1}^{j+1}}{t_{1,0}^jt_{0,0}^{j+1}}\psi_{0,0}^j = 0. \quad (5.3)$$

Equation (5.1) is invariant under the transformation defined as  $n \rightarrow 1 - m$ ,  $m \rightarrow 1 - n$ . Under this transformation the Lax pair (5.2) transforms to a Lax pair

$$y_{-1,0}^j = y_{0,0}^j + \frac{t_{-1,0}^{j+1}t_{0,0}^{j-1}}{t_{-1,0}^jt_{0,0}^j}y_{0,0}^{j-1}, \quad y_{0,-1}^j = \frac{t_{0,-1}^{j+1}t_{0,0}^j}{t_{0,0}^{j+1}t_{0,-1}^j}y_{0,0}^j - y_{0,0}^{j+1}, \quad (5.4)$$

and equation (5.3) transforms to an equation

$$y_{1,1}^j - \frac{t_{1,0}^jt_{0,1}^{j+1}}{t_{0,0}^{j+1}t_{1,1}^j}y_{1,0}^j - \frac{t_{1,0}^jt_{0,1}^j}{t_{0,0}^jt_{1,1}^j}y_{0,1}^j + \frac{t_{1,0}^jt_{0,1}^{j+1}}{t_{0,0}^{j+1}t_{1,1}^j}y_{0,0}^j = 0. \quad (5.5)$$

Put  $y_{0,0}^j = \frac{t_{-1,0}^{j+1}}{t_{0,0}^j}g_{0,0}^j$ , then Lax pair (5.4) transforms to a Lax pair

$$g_{-1,0}^j = \frac{t_{-1,0}^{j+1}t_{-1,0}^j}{t_{0,0}^jt_{-2,0}^{j+1}}(g_{0,0}^j + g_{0,0}^{j-1}), \quad g_{0,-1}^j = \frac{t_{0,-1}^{j+1}t_{-1,0}^j}{t_{0,0}^{j+1}t_{-1,-1}^j}g_{0,0}^j - \frac{t_{0,-1}^jt_{-1,0}^{j+2}}{t_{0,0}^{j+1}t_{-1,-1}^j}g_{0,0}^{j+1}, \quad (5.6)$$

and equation (5.5) transforms to an equation

$$g_{1,1}^j - g_{1,0}^j - \frac{t_{1,0}^jt_{-1,1}^{j+1}}{t_{0,1}^{j+1}t_{0,0}^j}g_{0,1}^j + \frac{t_{1,0}^jt_{-1,0}^{j+1}}{t_{0,0}^{j+1}t_{0,0}^j}g_{0,0}^j = 0. \quad (5.7)$$

Thus we have two different Lax pairs for the Hirota chain and consequently two families (5.3), (5.7) of linear discrete equations enumerated by  $j$ . Study the question when an equation from one family can be related, by a linear transformation, to an equation from the other family. To this end we evaluate the Laplace invariants of these equations (see [20]).

Recall that the Laplace invariants of a discrete hyperbolic type equation of the form:

$$f_{1,1} + a_{0,0}f_{1,0} + b_{0,0}f_{0,1} + c_{0,0}f_{0,0} = 0 \quad (5.8)$$

are given by (see [20])

$$K_1 = \frac{b_{0,0}c_{1,0}}{d_{0,0}a_{1,0}}, \quad K_2 = \frac{b_{0,1}c_{0,0}}{d_{0,0}a_{1,0}}. \quad (5.9)$$

By virtue of these formulas the invariants  $K_{1\psi}$ ,  $K_{2\psi}$  and  $K_{1g}$ ,  $K_{2g}$  of equations (5.3), (5.7) are, respectively,

$$K_{1\psi} = \frac{t_{1,0}^jt_{1,1}^jt_{0,1}^{j+1}t_{2,1}^{j+1}}{t_{0,0}^jt_{2,1}^jt_{1,1}^{j+1}t_{1,1}^{j+1}}, \quad K_{2\psi} = \frac{t_{1,0}^jt_{0,1}^j}{t_{1,1}^jt_{0,0}^j}, \quad (5.10)$$

$$K_{1g} = \frac{t_{0,0}^jt_{2,1}^jt_{0,0}^{j+1}t_{0,1}^{j+1}}{t_{1,0}^jt_{1,0}^jt_{-1,0}^{j+1}t_{1,1}^{j+1}}, \quad K_{2g} = \frac{t_{0,0}^{j+1}t_{-1,1}^{j+1}}{t_{-1,0}^{j+1}t_{0,1}^{j+1}}. \quad (5.11)$$



It is known that two linear hyperbolic type equations are related to one another by a linear transformation only if their corresponding Laplace invariants are equal. Evidently in generic case coincidence of the Laplace invariants generates two constraints on the field variables  $t^j = t^j(n, m)$ . Only for some special cases it gives only one constraint. We are interested in such special cases. For instance pair of equations  $K_{1\psi}(n, m, j) = K_{1g}(n+1, m, j-1)$ ,  $K_{2\psi}(n, m, j) = K_{2g}(n+1, m, j-1)$  is equivalent to the constraint

$$t_{1,0}^{j-1} = t_{0,1}^{j+1} \quad (5.12)$$

which is interpreted as a cutting off boundary condition for the chain (5.1). For simplicity we put  $j = 0$ , so the boundary condition becomes

$$t_{1,0}^{-1} = t_{0,1}^1. \quad (5.13)$$

**Remark 1.** Hirota equation (5.1) is compatible with the reduction of the type of parity

$$t_{m+i,k}^{-m} = t_{i,m+k}^m \quad (5.14)$$

and boundary condition (5.13) is a consequence of this reduction.

Following [21] we can construct a Lax pair for the reduced chain. Under the boundary condition (5.13) we have coincidence of the invariants

$$\begin{aligned} K_{1\psi}(n, m, j) &= K_{1g}(n+1, m, j-1), \\ K_{2\psi}(n, m, j) &= K_{2g}(n+1, m, j-1) \end{aligned} \quad (5.15)$$

and

$$\begin{aligned} K_{1g}(n, m, j) &= K_{1\psi}(n, m-1, j-1), \\ K_{2g}(n, m, j) &= K_{2\psi}(n, m-1, j-1) \end{aligned} \quad (5.16)$$

and we can relate the eigenfunctions

$$g_{0,1}^1 = \lambda \psi_{0,0}^0, \quad g_{1,0}^0 = \lambda \psi_{0,0}^1. \quad (5.17)$$

We study the finite reductions of the chain (5.1) on a finite interval  $N_L \leq j \leq N_R$ . The reduction is obtained by imposing the boundary conditions at the left end-point  $j = N_L$

$$t_{1,0}^{N_L-1} = t_{0,1}^{N_L+1} \quad (5.18)$$

and respectively at the right end-point  $j = N_R$

$$t_{0,1}^{N_R+1} = t_{1,0}^{N_R-1}. \quad (5.19)$$

First we concentrate on the left end-point. Due to the reasonings above the eigenfunctions should satisfy the following gluing conditions

$$\psi_{0,0}^{N_L-1} = \frac{1}{\lambda} g_{0,1}^{N_L}, \quad g_{0,0}^{N_L-1} = \lambda \psi_{-1,0}^{N_L}. \quad (5.20)$$

These conditions allow one to close the Lax equations at the left end-point

$$g_{-1,0}^{N_L} = \frac{t_{-1,0}^{N_L+1} t_{-1,0}^{N_L}}{t_{0,0}^{N_L} t_{-2,0}^{N_L+1}} (g_{0,0}^{N_L} + \lambda \psi_{-1,0}^{N_L}), \quad (5.21)$$

$$\psi_{0,1}^{N_L} = \psi_{0,0}^{N_L} + \frac{1}{\lambda} \frac{t_{0,1}^{N_L+1} t_{-1,1}^{N_L+1}}{t_{0,0}^{N_L} t_{0,1}^{N_L}} g_{0,1}^{N_L}. \quad (5.22)$$

From (5.2), (5.6) we have:

$$g_{0,-1}^{N_L} = \frac{t_{0,-1}^{N_L+1} t_{-1,0}^{N_L+1}}{t_{0,0}^{N_L+1} t_{-1,-1}^{N_L+1}} g_{0,0}^{N_L} - \frac{t_{0,-1}^{N_L} t_{-1,0}^{N_L+2}}{t_{0,0}^{N_L+1} t_{-1,-1}^{N_L+1}} g_{0,0}^{N_L+1}, \quad (5.23)$$

$$\psi_{1,0}^{N_L} = \frac{t_{1,0}^{N_L+1} t_{0,0}^{N_L}}{t_{0,0}^{N_L+1} t_{1,0}^{N_L}} \psi_{0,0}^{N_L} - \psi_{0,0}^{N_L+1}.$$

To derive similar equations at the point  $N_R$  we use the right end-point constraint (5.19), for which we have

$$g_{0,0}^{N_R} = \psi_{0,-1}^{N_R-1}, \quad \psi_{0,0}^{N_R} = g_{1,0}^{N_R-1}. \quad (5.24)$$

These conditions allow one to close the Lax equations at the right end-point

$$\psi_{1,0}^{N_R-1} = \frac{t_{0,0}^{N_R-1} t_{1,0}^{N_R}}{t_{1,0}^{N_R-1} t_{0,0}^{N_R}} \psi_{0,0}^{N_R-1} - g_{1,0}^{N_R-1}, \quad (5.25)$$

$$g_{0,-1}^{N_R-1} = \frac{t_{0,-1}^{N_R} t_{-1,0}^{N_R}}{t_{0,0}^{N_R} t_{-1,-1}^{N_R}} g_{0,0}^{N_R-1} - \frac{(t_{0,-1}^{N_R-1})^2}{t_{0,0}^{N_R} t_{-1,-1}^{N_R}} \psi_{0,-1}^{N_R-1}. \quad (5.26)$$

From (5.2), (5.6) we have:

$$g_{-1,0}^{N_R-1} = \frac{t_{-1,0}^{N_R} t_{-1,0}^{N_R-1}}{t_{0,0}^{N_R-1} t_{-2,0}^{N_R}} (g_{0,0}^{N_R-1} + g_{0,0}^{N_R-2}), \quad (5.27)$$

$$\psi_{0,1}^{N_R-1} = \psi_{0,0}^{N_R-1} + \frac{t_{0,1}^{N_R} t_{0,0}^{N_R-2}}{t_{0,0}^{N_R-1} t_{0,1}^{N_R-1}} \psi_{0,0}^{N_R-2}. \quad (5.28)$$

The Lax pair found above is not convenient to work with since the operators contain shifts in opposite directions. They can be rewritten in the usual form of Zakharov-Shabat system.

Shift of equation (5.21) forward with respect to the variable  $n$  brings it to the form

$$g_{1,0}^{N_L} = \frac{t_{-1,0}^{N_L+1} t_{1,0}^{N_L}}{t_{0,0}^{N_L} t_{0,0}^{N_L+1}} g_{0,0}^{N_L} - \lambda \psi_{0,0}^{N_L}, \quad (5.29)$$

$$g_{1,0}^j = \frac{t_{-1,0}^{j+1} t_{1,0}^j}{t_{0,0}^j t_{0,0}^{j+1}} g_{0,0}^j - g_{1,0}^{j-1} = \sum_{k=N_L}^j (-1)^{j-k} \frac{t_{1,0}^k t_{-1,0}^{k+1}}{t_{0,0}^k t_{0,0}^{k+1}} g_{0,0}^k + (-1)^{j+1} \lambda \psi_{0,0}^{N_L},$$

$$N_L + 1 \leq j \leq N_R - 1.$$

Shift of equation (5.26) forward with respect to the variable  $m$  brings it to the form

$$g_{0,1}^{N_R-1} = \frac{t_{0,1}^{N_R} t_{-1,0}^{N_R}}{t_{0,0}^{N_R} t_{-1,1}^{N_R}} g_{0,0}^{N_R-1} + \frac{(t_{0,0}^{N_R-1})^2}{t_{0,0}^{N_R} t_{-1,1}^{N_R}} \psi_{0,0}^{N_R-1}, \quad (5.30)$$

$$g_{0,1}^j = \frac{t_{-1,0}^{j+1} t_{0,1}^{j+1}}{t_{0,0}^{j+1} t_{-1,1}^{j+1}} g_{0,0}^j + \frac{t_{-1,0}^{j+1} t_{0,1}^{j+1}}{t_{0,0}^{j+1} t_{-1,1}^{j+1}} g_{0,1}^{j+1} = \frac{t_{-1,0}^{j+1} t_{0,1}^{j+1}}{t_{0,0}^{j+1} t_{-1,1}^{j+1}} g_{0,0}^j + \sum_{k=j+1}^{N_R-1} \frac{t_{0,0}^j t_{-1,0}^{k+1} t_{0,1}^{k+1}}{t_{-1,1}^{j+1} t_{0,0}^k t_{0,0}^{k+1}} g_{0,0}^k + \frac{t_{0,0}^j t_{0,0}^{N_R-1}}{t_{0,0}^{j+1} t_{0,0}^{N_R}} \psi_{0,0}^{N_R-1},$$

$$N_L \leq j \leq N_R - 1.$$

So we have the following system of linear equations

$$\psi_{1,0}^j = \frac{t_{1,0}^{j+1} t_{0,0}^j}{t_{0,0}^{j+1} t_{1,0}^j} \psi_{0,0}^j - \psi_{0,0}^{j+1}, \quad N_L \leq j \leq N_R - 2, \quad (5.31)$$

$$\psi_{1,0}^{N_R-1} = \frac{t_{0,0}^{N_R-1} t_{1,0}^{N_R}}{t_{1,0}^{N_R-1} t_{0,0}^{N_R}} \psi_{0,0}^{N_R-1} + \sum_{k=N_L}^{N_R-1} (-1)^{N_R-k} \frac{t_{1,0}^k t_{-1,0}^{k+1}}{t_{0,0}^k t_{0,0}^{k+1}} g_{0,0}^k + (-1)^{N_R-1} \lambda \psi_{0,0}^{N_L}, \quad (5.32)$$

$$\psi_{0,1}^{N_L} = \psi_{0,0}^{N_L} + \frac{1}{\lambda} \frac{t_{0,1}^{N_L+1} t_{0,0}^{N_R-1}}{t_{0,1}^{N_L} t_{0,0}^{N_R}} \psi_{0,0}^{N_R-1} + \sum_{j=N_L}^{N_R-1} \frac{1}{\lambda} \frac{t_{0,1}^{N_L+1} t_{0,1}^{k+1} t_{-1,0}^{k+1}}{t_{0,1}^{N_L} t_{0,0}^k t_{0,0}^{k+1}} g_{0,0}^k, \quad (5.33)$$

$$\psi_{0,1}^j = \psi_{0,0}^j + \frac{t_{0,1}^j t_{0,0}^{j-2}}{t_{0,0}^{j-1} t_{0,1}^{j-1}} \psi_{0,0}^{j-1}, \quad N_L + 1 \leq j \leq N_R - 1, \quad (5.34)$$

$$g_{1,0}^j = \sum_{k=N_L}^j (-1)^{j-k} \frac{t_{1,0}^k t_{-1,0}^{k+1}}{t_{0,0}^k t_{0,0}^{k+1}} g_{0,0}^k + (-1)^{j+1} \lambda \psi_{0,0}^{N_L}, \quad N_L \leq j \leq N_R - 1, \quad (5.35)$$

$$g_{0,1}^j = \frac{t_{-1,0}^{j+1} t_{0,1}^{j+1}}{t_{0,0}^{j+1} t_{-1,1}^{j+1}} g_{0,0}^j + \sum_{k=j+1}^{N_R-1} \frac{t_{0,0}^j t_{-1,0}^{k+1} t_{0,1}^{k+1}}{t_{-1,1}^{j+1} t_{0,0}^k t_{0,0}^{k+1}} g_{0,0}^k + \frac{t_{0,0}^j t_{0,0}^{N_R-1}}{t_{0,0}^{j+1} t_{0,0}^{N_R}} \psi_{0,0}^{N_R-1}, \quad N_L \leq j \leq N_R - 1. \quad (5.36)$$

In the next subsection we gather these equations to a matrix form.

## 5.1 Integrable systems corresponding to the Cartan matrices $D_N^{(2)}$ , $A_1^{(1)}$

Imposing of non-degenerate boundary conditions (5.18), (5.19) at both end-points  $N_L = 0$  and  $N_R = N$  leads to the system corresponding to the matrix  $D_{N+1}^{(2)}$ ,  $N \geq 2$

$$\begin{aligned} t_{0,0}^0 t_{1,1}^0 - t_{1,0}^0 t_{0,1}^0 &= (t_{0,1}^1)^2, \\ t_{0,0}^j t_{1,1}^j - t_{1,0}^j t_{0,1}^j &= t_{1,0}^{j-1} t_{0,1}^{j+1}, \quad 1 \leq j \leq N-1, \\ t_{0,0}^N t_{1,1}^N - t_{1,0}^N t_{0,1}^N &= (t_{1,0}^{N-1})^2. \end{aligned} \quad (5.37)$$

Write the set of equations (5.31)-(5.36) in the form of a Lax pair for system (5.37). Denote the eigenvector as follow  $P = (\psi^0, \psi^1, \dots, \psi^{N-1}, g^0, g^1, \dots, g^{N-1})^T$ . Introduce  $2N \times 2N$  matrices

$$A = \begin{pmatrix} \frac{t_{1,0}^1 t_{0,0}^0}{t_{0,0}^1 t_{1,0}^0} & -1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & -1 & \dots & 0 & 0 & \dots & 0 \\ & & & \dots & & & & \\ (-1)^{N-1} \lambda & 0 & \dots & \frac{t_{1,0}^N t_{0,0}^{N-1}}{t_{0,0}^N t_{1,0}^{N-1}} & (-1)^N \frac{t_{1,0}^0 t_{-1,0}^1}{t_{0,0}^0 t_{0,0}^1} & (-1)^{N-1} \frac{t_{1,0}^1 t_{-1,0}^2}{t_{0,0}^1 t_{0,0}^2} & \dots & -\frac{t_{1,0}^{N-1} t_{-1,0}^N}{t_{0,0}^{N-1} t_{0,0}^N} \\ -\lambda & 0 & \dots & 0 & \frac{t_{1,0}^0 t_{-1,0}^1}{t_{0,0}^0 t_{0,0}^1} & 0 & \dots & 0 \\ \lambda & 0 & \dots & 0 & -\frac{t_{1,0}^0 t_{-1,0}^1}{t_{0,0}^0 t_{0,0}^1} & \frac{t_{1,0}^1 t_{-1,0}^2}{t_{0,0}^1 t_{0,0}^2} & \dots & 0 \\ & & & \dots & & & & \\ (-1)^{i+1} \lambda & 0 & \dots & 0 & \dots & (-1)^{i-j} \frac{t_{1,0}^j t_{-1,0}^{j+1}}{t_{0,0}^j t_{0,0}^{j+1}} & \dots & 0 \\ & & & \dots & & & & \\ (-1)^N \lambda & 0 & \dots & 0 & (-1)^{N-1} \frac{t_{1,0}^0 t_{-1,0}^1}{t_{0,0}^0 t_{0,0}^1} & (-1)^N \frac{t_{1,0}^1 t_{-1,0}^2}{t_{0,0}^1 t_{0,0}^2} & \dots & \frac{t_{1,0}^{N-1} t_{-1,0}^N}{t_{0,0}^{N-1} t_{0,0}^N} \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & \frac{1}{\lambda} \frac{t_{0,1}^1 t_{0,0}^{N-1}}{t_{0,0}^0 t_{0,1}^N} & \frac{1}{\lambda} \frac{(t_{0,1}^1)^2 t_{-1,0}^1}{t_{0,0}^0 t_{0,1}^0 t_{0,0}^1} & \frac{1}{\lambda} \frac{t_{0,1}^1 t_{-1,0}^2 t_{0,1}^2}{t_{0,1}^0 t_{0,0}^1 t_{0,0}^2} & \cdots & \frac{1}{\lambda} \frac{t_{0,1}^1 t_{-1,0}^N t_{0,1}^N}{t_{0,1}^0 t_{0,0}^{N-1} t_{0,0}^N} \\ \frac{t_{0,0}^0 t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{t_{0,0}^1 t_{0,1}^3}{t_{0,0}^2 t_{0,1}^2} & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & & & \cdots & & & & & \\ 0 & 0 & \cdots & \frac{t_{0,0}^{N-2} t_{0,1}^N}{t_{0,0}^{N-1} t_{0,1}^{N-1}} & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{t_{0,0}^0 t_{0,0}^{N-1}}{t_{-1,1}^1 t_{0,0}^N} & \frac{t_{0,1}^1 t_{-1,0}^1}{t_{0,0}^1 t_{-1,1}^1} & \frac{t_{0,0}^0 t_{-1,0}^2 t_{0,1}^2}{t_{-1,1}^1 t_{0,0}^1 t_{0,0}^2} & \cdots & \frac{t_{0,0}^0 t_{-1,0}^N t_{0,1}^N}{t_{-1,1}^1 t_{0,0}^{N-1} t_{0,0}^N} \\ 0 & 0 & \cdots & 0 & \frac{t_{0,0}^1 t_{0,0}^{N-1}}{t_{-1,1}^2 t_{0,0}^N} & 0 & \frac{t_{-1,0}^2 t_{0,1}^2}{t_{0,0}^2 t_{-1,1}^1} & \cdots & \frac{t_{0,0}^1 t_{-1,0}^N t_{0,1}^N}{t_{-1,1}^2 t_{0,0}^{N-1} t_{0,0}^N} \\ & & & \cdots & & & & & \\ 0 & 0 & \cdots & 0 & \frac{(t_{0,0}^{N-1})^2}{t_{-1,1}^N t_{0,0}^N} & 0 & 0 & \cdots & \frac{t_{-1,0}^N t_{0,1}^N}{t_{0,0}^N t_{-1,1}^N} \end{pmatrix}.$$

It is straightforward to check that the compatibility condition of the equations

$$P_{1,0} = AP, \quad P_{0,1} = BP \quad (5.38)$$

is equivalent the system (5.37).

**Example 1** Consider a particular case when  $N_L = 0$  and  $N_R = 1$

$$\begin{aligned} t_{0,0}^0 t_{1,1}^0 - t_{1,0}^0 t_{0,1}^0 &= (t_{0,1}^1)^2, \\ t_{0,0}^1 t_{1,1}^1 - t_{1,0}^1 t_{0,1}^1 &= (t_{1,0}^0)^2. \end{aligned} \quad (5.39)$$

The system corresponds to algebra  $A_1^{(1)}$ . Its Lax pair is given by (5.38) with the matrices

$$A = \begin{pmatrix} \frac{t_{1,0}^1 t_{0,0}^0}{t_{0,0}^1 t_{1,0}^0} + \lambda & -\frac{t_{1,0}^0 t_{-1,0}^1}{t_{0,0}^0 t_{0,0}^1} \\ -\lambda & \frac{t_{1,0}^0 t_{-1,0}^1}{t_{0,0}^0 t_{0,0}^1} \end{pmatrix}, \quad B = \begin{pmatrix} 1 + \frac{1}{\lambda} \frac{t_{0,1}^1 t_{0,0}^0}{t_{0,1}^0 t_{0,0}^1} & \frac{1}{\lambda} \frac{(t_{0,1}^1)^2 t_{-1,0}^1}{t_{0,0}^0 t_{0,1}^0 t_{0,0}^1} \\ \frac{(t_{0,0}^0)^2}{t_{0,0}^1 t_{-1,1}^1} & \frac{t_{0,1}^1 t_{-1,0}^1}{t_{0,0}^1 t_{-1,1}^1} \end{pmatrix}.$$

**Example 2** If one imposes condition (5.18) for  $N_L = 0$  and (5.19) for  $N_R = 2$ , then

$$\begin{aligned} t_{0,0}^0 t_{1,1}^0 - t_{1,0}^0 t_{0,1}^0 &= (t_{0,1}^1)^2, \\ t_{0,0}^1 t_{1,1}^1 - t_{1,0}^1 t_{0,1}^1 &= t_{1,0}^0 t_{0,1}^2, \\ t_{0,0}^2 t_{1,1}^2 - t_{1,0}^2 t_{0,1}^2 &= (t_{1,0}^1)^2. \end{aligned} \quad (5.40)$$

The Lax pair is of the form (5.38) where  $A$  and  $B$  are  $4 \times 4$  matrices

$$A = \begin{pmatrix} \frac{t_{1,0}^1 t_{0,0}^0}{t_{0,0}^1 t_{1,0}^0} & -1 & 0 & 0 \\ -\lambda & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & \frac{t_{1,0}^0 t_{-1,0}^1}{t_{0,0}^0 t_{0,0}^1} & -\frac{t_{1,0}^1 t_{-1,0}^2}{t_{0,0}^1 t_{0,0}^2} \\ -\lambda & 0 & \frac{t_{1,0}^0 t_{-1,0}^1}{t_{0,0}^0 t_{0,0}^1} & 0 \\ \lambda & 0 & -\frac{t_{1,0}^0 t_{-1,0}^1}{t_{0,0}^0 t_{0,0}^1} & \frac{t_{1,0}^1 t_{-1,0}^2}{t_{0,0}^1 t_{0,0}^2} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \frac{1}{\lambda} \frac{t_{0,1}^1 t_{0,0}^0}{t_{0,1}^0 t_{0,0}^1} & \frac{1}{\lambda} \frac{(t_{0,1}^1)^2 t_{-1,0}^1}{t_{0,0}^0 t_{0,1}^0 t_{0,0}^1} & \frac{1}{\lambda} \frac{t_{0,1}^1 t_{-1,0}^2 t_{0,1}^2}{t_{0,1}^0 t_{0,0}^1 t_{0,0}^2} \\ \frac{t_{0,0}^0 t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 & 0 & 0 \\ 0 & \frac{t_{0,0}^0 t_{0,1}^0}{t_{-1,1}^1 t_{0,0}^2} & \frac{t_{0,1}^1 t_{-1,0}^1}{t_{0,0}^1 t_{-1,1}^1} & \frac{t_{0,0}^0 t_{-1,0}^2 t_{0,1}^2}{t_{-1,1}^1 t_{0,0}^1 t_{0,0}^2} \\ 0 & \frac{(t_{0,0}^0)^2}{t_{-1,1}^2 t_{0,0}^2} & 0 & \frac{t_{-1,0}^2 t_{0,1}^2}{t_{0,0}^2 t_{-1,1}^1} \end{pmatrix}.$$

## 5.2 Integrable systems corresponding to the Cartan matrices $A_N$

In addition to boundary condition (5.13) we introduce the degenerate boundary conditions

$$t^{N_L-1} = 0, \quad t^{N_L} = 1 \quad (5.41)$$

at the left end-point and

$$t^{N_R+1} = 0, \quad t^{N_R} = 1 \quad (5.42)$$

at the right end-point. The degenerate boundary conditions imply that the corresponding eigenfunctions are zero:  $g^{N_L-1} = 0$  and  $\psi^{N_R} = 0$ .

In order to obtain the system corresponding to the Cartan matrix  $A_{N-1}$  we cut off the Hirota chain by imposing degenerate boundary conditions (5.41) at the point  $N_L = 0$  and (5.42) at  $N_R = N$ . The resulting reduction is as follows

$$\begin{aligned} t_{0,0}^1 t_{1,1}^1 - t_{1,0}^1 t_{0,1}^1 &= t_{0,1}^2, \\ t_{0,0}^j t_{1,1}^j - t_{1,0}^j t_{0,1}^j &= t_{1,0}^{j-1} t_{0,1}^{j+1}, \quad 2 \leq j \leq N-2, \\ t_{0,0}^{N-1} t_{1,1}^{N-1} - t_{1,0}^{N-1} t_{0,1}^{N-1} &= t_{1,0}^{N-2}. \end{aligned} \quad (5.43)$$

In this case our algorithm gives the Lax pair found years ago in [13]. In order to formulate it introduce the eigenvector  $P = (\psi^0, \psi^1, \dots, \psi^{N-1})^T$  and  $N \times N$  matrices

$$U = \begin{pmatrix} \frac{t_{1,0}^1}{t_{0,0}^1} & -1 & 0 & 0 & \dots & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & -1 & 0 & \dots & 0 \\ 0 & 0 & \frac{t_{1,0}^3 t_{0,0}^2}{t_{0,0}^3 t_{1,0}^2} & -1 & \dots & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & \frac{t_{0,0}^{N-1}}{t_{1,0}^{N-1}} \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 & 0 & \dots & 0 & 0 \\ 0 & \frac{t_{0,1}^3 t_{0,0}^1}{t_{0,0}^2 t_{0,1}^2} & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & \dots & 0 & \frac{t_{0,0}^{N-2}}{t_{0,0}^{N-1} t_{0,1}^{N-1}} & 1 \end{pmatrix} \quad (5.44)$$

It is easy to check that the compatibility condition of the equations

$$P_{1,0} = UP, \quad P_{0,1} = VP. \quad (5.45)$$

leads to the system (5.43).

### 5.3 Integrable systems corresponding to the Cartan matrices $B_N$

We impose boundary condition (5.41) for  $N_L = 0$  and (5.19) for  $N_R = N$ . The resulting reduction is as follows

$$\begin{aligned} t_{0,0}^1 t_{1,1}^1 - t_{1,0}^1 t_{0,1}^1 &= t_{0,1}^2, \\ t_{0,0}^j t_{1,1}^j - t_{1,0}^j t_{0,1}^j &= t_{1,0}^{j-1} t_{0,1}^{j+1}, \quad 2 \leq j \leq N-1, \\ t_{0,0}^N t_{1,1}^N - t_{1,0}^N t_{0,1}^N &= (t_{1,0}^{N-1})^2. \end{aligned} \quad (5.46)$$

System (5.46) can be rewritten in form of (1.3) by changing the variables  $t^j = e^{-u^j}$ . It corresponds to the algebra  $B_N$ . The system admits a Lax pair. Let us denote  $P = (\psi^0, \psi^1, \dots, \psi^{N-1}, g^0, g^1, \dots, g^{N-1})^T$  and introduce  $2N \times 2N$  matrices

$$A = \begin{pmatrix} \frac{t_{1,0}^1}{t_{0,0}^1} & -1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & -1 & \dots & 0 & 0 & \dots & 0 \\ & & & \dots & & & & \\ 0 & 0 & \dots & \frac{t_{1,0}^N t_{0,0}^{N-1}}{t_{0,0}^N t_{1,0}^{N-1}} & (-1)^N \frac{t_{1,0}^1}{t_{0,0}^1} & (-1)^{N-1} \frac{t_{1,0}^1 t_{0,0}^2}{t_{0,0}^1 t_{1,0}^2} & \dots & -\frac{t_{1,0}^{N-1} t_{0,0}^N}{t_{0,0}^{N-1} t_{1,0}^N} \\ 0 & 0 & \dots & 0 & \frac{t_{1,0}^1}{t_{0,0}^1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -\frac{t_{1,0}^1}{t_{0,0}^1} & \frac{t_{1,0}^1 t_{0,0}^2}{t_{0,0}^1 t_{1,0}^2} & \dots & 0 \\ & & & \dots & & & & \\ 0 & 0 & \dots & 0 & (-1)^{N-1} \frac{t_{1,0}^1}{t_{0,0}^1} & (-1)^N \frac{t_{1,0}^1 t_{0,0}^2}{t_{0,0}^1 t_{1,0}^2} & \dots & \frac{t_{1,0}^{N-1} t_{0,0}^N}{t_{0,0}^{N-1} t_{1,0}^N} \end{pmatrix}, \quad (5.47)$$

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \frac{t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{t_{0,0}^1 t_{0,1}^3}{t_{0,0}^2 t_{0,1}^2} & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ & & & \dots & & & & & \\ 0 & 0 & \dots & \frac{t_{0,0}^{N-2} t_{0,1}^N}{t_{0,0}^{N-1} t_{0,1}^{N-1}} & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{t_{0,0}^{N-1}}{t_{1,0}^1 t_{0,0}^N} & \frac{t_{0,1}^1 t_{1,0}^1}{t_{0,0}^1 t_{1,0}^1} & \frac{t_{1,0}^2 t_{0,1}^2}{t_{1,0}^1 t_{0,0}^2 t_{0,1}^2} & \dots & \frac{t_{1,0}^N t_{0,1}^N}{t_{1,0}^1 t_{0,0}^{N-1} t_{0,1}^N} \\ 0 & 0 & \dots & 0 & \frac{t_{0,0}^1 t_{0,0}^{N-1}}{t_{1,0}^2 t_{0,0}^N} & 0 & \frac{t_{1,0}^2 t_{0,1}^2}{t_{0,0}^2 t_{1,0}^1} & \dots & \frac{t_{0,0}^1 t_{1,0}^{N-1} t_{0,1}^N}{t_{1,0}^2 t_{0,0}^{N-1} t_{0,1}^N} \\ & & & \dots & & & & & \\ 0 & 0 & \dots & 0 & \frac{(t_{0,0}^{N-1})^2}{t_{1,0}^1 t_{0,0}^N} & 0 & 0 & \dots & \frac{t_{1,0}^N t_{0,1}^N}{t_{0,0}^N t_{1,0}^1} \end{pmatrix} \quad (5.48)$$

Then according to our general scheme the compatibility condition of the equations

$$P_{1,0} = AP, \quad P_{0,1} = BP. \quad (5.49)$$

leads to the system (5.46).

**Remark 2.** The system  $B_N$  can be obtained from the system  $A_{2N-1}$  by imposing the cutting off constraint of the form

$$t_{0,1}^{N+1} = t_{1,0}^{N-1}, \quad (5.50)$$

(see Remark 1 above).

## 6 Method of finding integrals from Lax representation for the systems corresponding to the Cartan matrices $A_N$ , $B_N$

For any integer  $N \geq 2$  system (5.43), corresponding  $A_N$ , admits a complete set of independent integrals consisting of  $N$  independent  $m$ -integrals and  $N$  independent  $n$ -integrals. Demonstrate an algorithm of finding them by using Zakharov-Shabat operators. Begin with  $m$ -integrals. Due to the definition we have two different expression for the shifted eigenvector  $P_{k,1}$ ,  $k \geq 1$

$$P_{k,1} = V_{k,0}U_{k-1,0}U_{k-2,0} \dots U_{1,0}U_{0,0}P,$$

and similarly

$$P_{k,1} = U_{k-1,1}U_{k-2,1} \dots U_{1,1}U_{0,1}V_{0,0}P.$$

Comparison of these two formulas yields

$$D_m(U_{k-1,0}U_{k-2,0} \dots U_{1,0}U_{0,0}) = V_{k,0}U_{k-1,0}U_{k-2,0} \dots U_{1,0}U_{0,0}V_{0,0}^{-1}. \quad (6.1)$$

Due to the triangular structure of the matrices  $V_{k,0}$  and  $V_{0,0}$  the map converting any upper triangular matrix  $X$  to a matrix  $\bar{X} = V_{k,0}XV_{0,0}^{-1}$  leaves unchanged the element of the matrix  $X$  located at the right upper corner:  $(X)_{1,N} = \bar{X}_{1,N}$ . Thus the corresponding element, denote it through  $I_k$ , of the upper triangular matrix  $U_{k-1,0}U_{k-2,0} \dots U_{1,0}U_{0,0}$  is an  $m$ -integral. It can be proved that integrals  $I_N, I_{N+1}, \dots, I_{2N}$  constitute a complete set of  $m$ -integrals. In a similar way one can find integrals in the other direction.

Let us illustrate the statement above with the following



**Example 3** Consider the system (5.43) for  $N = 2$

$$\begin{aligned} t_{0,0}^1 t_{1,1}^1 - t_{1,0}^1 t_{0,1}^1 &= t_{0,1}^2, \\ t_{0,0}^2 t_{1,1}^2 - t_{1,0}^2 t_{0,1}^2 &= t_{1,0}^1. \end{aligned} \quad (6.2)$$

Recall its Lax pair

$$P_{1,0} = UP, \quad P_{0,1} = VP, \quad (6.3)$$

where  $P = (\psi^0, \psi^1, \psi^2)^T$  and

$$U = \begin{pmatrix} \frac{t_{1,0}^1}{t_{0,0}^1} & -1 & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & -1 \\ 0 & 0 & \frac{t_{0,0}^2}{t_{1,0}^2} \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 \\ \frac{t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 & 0 \\ 0 & \frac{t_{0,0}^1}{t_{0,0}^2 t_{0,1}^2} & 1 \end{pmatrix}. \quad (6.4)$$

Evaluate the elements at the right upper corner for the following two products  $U_{2,0}U_{1,0}U_{0,0}$  and  $U_{3,0}U_{2,0}U_{1,0}U_{0,0}$  and find two independent  $m$ -integrals

$$I^1 = \frac{t_3^1}{t_2^1} + \frac{t_2^1 t_2^2}{t_2^1 t_1^2} + \frac{t_0^2}{t_1^2}, \quad I^2 = \frac{t_4^1}{t_3^1} I^1 + \frac{t_3^1 t_1^1 (t_2^2)^2}{(t_2^1)^2 t_3^2 t_1^2} + \frac{t_3^1 t_2^2 t_0^2}{t_2^1 t_3^2 t_1^2} + \frac{t_0^2}{t_2^2},$$

where the second index for the variables  $t^1, t^2$  is omitted.

In a similar way integrals for the system (5.46), corresponding to the algebra  $B_N$ , are constructed.

Let us consider a matrix  $\Phi$  of dimension  $2N \times 2N$ :

$$\Phi = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \quad (6.5)$$

where  $E_{11}$  is the unity matrix of dimension  $N$ ,  $E_{12}, E_{21}$  are matrices of dimension  $N$  with zero entries,  $E_{22}$  is a matrix of dimension  $N$  with the unity adverse diagonal. By an automorphism

$X \rightarrow \Phi X \Phi^{-1}$  the matrices (5.47) and (5.48) are transformed to the triangular matrices

$$\bar{A} = \begin{pmatrix} \frac{t_{1,0}^1}{t_{0,0}^1} & -1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & -1 & \dots & 0 & 0 & \dots & 0 \\ & & & \dots & & & & \\ 0 & 0 & \dots & \frac{t_{1,0}^N t_{0,0}^{N-1}}{t_{0,0}^N t_{1,0}^{N-1}} & -\frac{t_{1,0}^{N-1} t_{-1,0}^N}{t_{0,0}^{N-1} t_{0,0}^N} & \frac{t_{1,0}^{N-2} t_{-1,0}^{N-1}}{t_{0,0}^{N-2} t_{0,0}^{N-1}} & \dots & (-1)^N \frac{t_{-1,0}^1}{t_{0,0}^1} \\ 0 & 0 & \dots & 0 & \frac{t_{1,0}^{N-1} t_{-1,0}^N}{t_{0,0}^{N-1} t_{0,0}^N} & -\frac{t_{1,0}^{N-2} t_{-1,0}^{N-1}}{t_{0,0}^{N-2} t_{0,0}^{N-1}} & \dots & (-1)^{N-1} \frac{t_{-1,0}^1}{t_{0,0}^1} \\ 0 & 0 & \dots & 0 & 0 & \frac{t_{1,0}^{N-2} t_{-1,0}^{N-1}}{t_{0,0}^{N-2} t_{0,0}^{N-1}} & \dots & (-1)^N \frac{t_{-1,0}^1}{t_{0,0}^1} \\ & & & \dots & & & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & \frac{t_{-1,0}^1}{t_{0,0}^1} \end{pmatrix},$$

$$\bar{B} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \frac{t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ & & & \dots & & & & & \\ 0 & 0 & \dots & \frac{t_{0,0}^{N-2} t_{0,1}^N}{t_{0,0}^{N-1} t_{0,1}^{N-1}} & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{t_{0,0}^{N-1}}{t_{-1,1}^1 t_{0,0}^N} & \frac{t_{-1,0}^N t_{0,1}^N}{t_{0,0}^N t_{-1,1}^N} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{t_{0,0}^{N-1}}{t_{-1,1}^1 t_{0,0}^N} & \frac{t_{0,0}^{N-2} t_{-1,0}^N t_{0,1}^N}{t_{-1,1}^1 t_{0,0}^{N-1} t_{0,0}^N} & \frac{t_{-1,0}^{N-1} t_{0,1}^{N-1}}{t_{0,0}^{N-1} t_{-1,1}^N} & \dots & 0 \\ & & & \dots & & & & & \\ 0 & 0 & \dots & 0 & \frac{(t_{0,0}^{N-1})^2}{t_{-1,1}^N t_{0,0}^N} & \frac{t_{-1,0}^N t_{0,1}^N}{t_{-1,1}^1 t_{0,0}^{N-1} t_{0,0}^N} & \frac{t_{-1,0}^{N-1} t_{0,1}^{N-1}}{t_{-1,1}^1 t_{0,0}^{N-2} t_{0,0}^{N-1}} & \dots & \frac{t_{0,1}^1 t_{-1,0}^1}{t_{0,0}^1 t_{-1,1}^1} \end{pmatrix}. \quad (6.6)$$

By an automorphism acting as follows  $X \rightarrow F^{-1} X F$  matrix (6.6) is transformed to a lower triangular matrix for which all diagonal entries are equal to the unity. Here  $F$  is a  $2N \times 2N$  matrix of the form

$$F = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & F_{22} \end{pmatrix}, \quad (6.7)$$

$F_{22}$  is a diagonal matrix such that  $F_{22} = \text{diag} \left( \frac{t_{0,0}^N}{t_{-1,0}^N}, \frac{t_{0,0}^{N-1}}{t_{-1,0}^{N-1}}, \dots, \frac{t_{0,0}^1}{t_{-1,0}^1} \right)$  and  $E_{i,j}$  are defined in (6.5). Reasonings similar to that of the case  $A_N$  allow one to derive the integrals.

**Example 4** Consider the system (5.46) for  $N = 2$

$$\begin{aligned} t_{0,0}^1 t_{1,1}^1 - t_{1,0}^1 t_{0,1}^1 &= t_{0,1}^2, \\ t_{0,0}^2 t_{1,1}^2 - t_{1,0}^2 t_{0,1}^2 &= (t_{1,0}^1)^2. \end{aligned} \quad (6.8)$$

Recall its Lax pair

$$P_{1,0} = AP, \quad P_{0,1} = BP, \quad (6.9)$$

where  $P = (\psi^0, \psi^1, g^0, g^1)^T$  and

$$A = \begin{pmatrix} \frac{t_{1,0}^1}{t_{0,0}^1} & -1 & 0 & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & \frac{t_{1,0}^1}{t_{0,0}^1} & -\frac{t_{1,0}^1 t_{-1,0}^2}{t_{0,0}^1 t_{0,0}^2} \\ 0 & 0 & \frac{t_{1,0}^1}{t_{0,0}^1} & 0 \\ 0 & 0 & -\frac{t_{1,0}^1}{t_{0,0}^1} & \frac{t_{1,0}^1 t_{-1,0}^2}{t_{0,0}^1 t_{0,0}^2} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 & 0 & 0 \\ 0 & \frac{t_{0,0}^1}{t_{-1,1}^1 t_{0,0}^2} & \frac{t_{0,1}^1 t_{-1,0}^1}{t_{0,0}^1 t_{-1,1}^1} & \frac{t_{-1,0}^2 t_{0,1}^2}{t_{-1,1}^1 t_{0,0}^1 t_{0,0}^2} \\ 0 & \frac{(t_{0,0}^1)^2}{t_{-1,1}^1 t_{0,0}^2} & 0 & \frac{t_{-1,0}^2 t_{0,1}^2}{t_{0,0}^2 t_{-1,1}^1} \end{pmatrix}. \quad (6.10)$$

By an automorphism  $X \rightarrow \Phi X \Phi^{-1}$  transform the matrices  $A$  and  $B$  to the triangular matrices

$$\bar{A} = \begin{pmatrix} \frac{t_{1,0}^1}{t_{0,0}^1} & -1 & 0 & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & -\frac{t_{1,0}^1 t_{-1,0}^2}{t_{0,0}^1 t_{0,0}^2} & \frac{t_{1,0}^1}{t_{0,0}^1} \\ 0 & 0 & \frac{t_{1,0}^1 t_{-1,0}^2}{t_{0,0}^1 t_{0,0}^2} & -\frac{t_{1,0}^1}{t_{0,0}^1} \\ 0 & 0 & 0 & \frac{t_{1,0}^1}{t_{0,0}^1} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 & 0 & 0 \\ 0 & \frac{(t_{0,0}^1)^2}{t_{-1,1}^1 t_{0,0}^2} & \frac{t_{-1,0}^2 t_{0,1}^2}{t_{0,0}^2 t_{-1,1}^1} & 0 \\ 0 & \frac{t_{0,0}^1}{t_{-1,1}^1 t_{0,0}^2} & \frac{t_{-1,0}^2 t_{0,1}^2}{t_{-1,1}^1 t_{0,0}^1 t_{0,0}^2} & \frac{t_{-1,0}^1 t_{0,1}^1}{t_{-1,1}^1 t_{0,0}^1} \end{pmatrix}. \quad (6.11)$$

By automorphism  $X \rightarrow F^{-1} X F$  transform the matrices (6.11) to the triangular matrices

$$\hat{A} = \begin{pmatrix} \frac{t_{1,0}^1}{t_{0,0}^1} & -1 & 0 & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & -\frac{t_{1,0}^1}{t_{0,0}^1} & 1 \\ 0 & 0 & \frac{t_{1,0}^1 t_{0,0}^2}{t_{0,0}^1 t_{1,0}^2} & -\frac{t_{0,0}^2}{t_{1,0}^2} \\ 0 & 0 & 0 & \frac{t_{0,0}^1}{t_{1,0}^1} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 & 0 & 0 \\ 0 & \frac{(t_{0,0}^1)^2}{t_{0,0}^2 t_{0,1}^2} & 1 & 0 \\ 0 & \frac{t_{0,0}^1}{t_{0,1}^1 t_{0,0}^2} & \frac{t_{0,1}^2}{t_{0,0}^1 t_{0,1}^1} & 1 \end{pmatrix}. \quad (6.12)$$

Evaluate the elements at the right upper corner for the following products  $\hat{A}_{2,0} \hat{A}_{1,0} \hat{A}_{0,0}$  and  $\hat{A}_{3,0} \hat{A}_{2,0} \hat{A}_{1,0} \hat{A}_{0,0}$  and find two independent  $m$ -integrals

$$I^1 = -\frac{t_3^1}{t_2^1} - \frac{t_2^1 t_0^2}{t_2^1 t_1^2} - \frac{t_1^1 t_2^2}{t_2^1 t_1^2} - \frac{t_0^1}{t_1^1}, \quad I^2 = -\frac{t_4^1}{t_2^1} - \frac{t_1^1 t_2^2 t_4^1}{t_2^1 t_1^2 t_3^1} - \frac{t_1^1 t_3^2}{t_3^1 t_1^2} - \frac{t_2^1 t_0^2 t_4^1}{t_1^1 t_1^2 t_3^1} - \frac{(t_2^1)^2 t_0^2 t_3^2}{t_1^1 t_1^2 t_3^1 t_2^2} - \frac{t_3^1 t_0^2}{t_1^1 t_2^2} - \frac{t_0^1 t_4^1}{t_1^1 t_3^1} - \frac{t_0^1 t_2^2 t_3^1}{t_1^1 t_1^2 t_3^1} - \frac{t_0^1 t_3^1 t_1^2}{t_1^1 t_1^2 t_1^2} - \frac{t_0^1}{t_2^2}. \quad (6.13)$$

Since the integral  $I^2$  is too complicated one can choose a more simple one

$$\bar{I}^2 = I^2 + I^1 D_n I^1 = \frac{t_0^2}{t_1^2} + \frac{(t_1^1)^2 t_2^2}{(t_2^1)^2 t_1^2} + 2 \frac{t_3^1 t_1^1}{(t_2^1)^2} + \frac{(t_3^1)^2 t_1^2}{(t_2^1)^2 t_2^2} + \frac{t_3^2}{t_2^2}. \quad (6.14)$$

In the integrals above the second index for the variables  $t^1, t^2$  is omitted, because its values is zero for all considered variables.

## 7 Periodic boundary conditions

In this section we discuss briefly the well known periodically closed reduction of the Hirota chain (see for more details and the references [32]). Close the chain (5.1) by imposing the periodical boundary conditions

$$t^{-1} = t^N, \quad t^{N+1} = t^0. \quad (7.1)$$

Close the Lax pair (5.2) by setting the conditions on the eigenfunction

$$\psi^{-1} = \lambda \psi^N, \quad \psi^{N+1} = \frac{1}{\lambda} \psi^0. \quad (7.2)$$

As a result we get a finite system of the form

$$\begin{aligned} t_{0,0}^0 t_{1,1}^0 - t_{1,0}^0 t_{0,1}^0 &= t_{1,0}^N t_{0,1}^1, \\ t_{0,0}^j t_{1,1}^j - t_{1,0}^j t_{0,1}^j &= t_{1,0}^{j-1} t_{0,1}^{j+1}, \quad 1 \leq j \leq N-1, \\ t_{0,0}^N t_{1,1}^N - t_{1,0}^N t_{0,1}^N &= t_{1,0}^0 t_{0,1}^1 \end{aligned} \quad (7.3)$$

which is closely connected with the Cartan matrix  $A_N^{(1)}$ .

Boundary conditions (7.2) reduce the sequence of linear discrete equations (5.2) to a Lax pair for the reduced system (7.3). Introduce the eigenvector  $P = (\psi^0, \psi^1, \dots, \psi^N)^T$  and  $N+1 \times N+1$  matrices

$$U = \begin{pmatrix} \frac{t_{0,0}^0 t_{1,0}^1}{t_{1,0}^0 t_{0,0}^1} & -1 & 0 & \dots & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & -1 & \dots & 0 \\ 0 & 0 & \frac{t_{1,0}^3 t_{0,0}^2}{t_{0,0}^3 t_{1,0}^2} & \dots & 0 \\ & & & \dots & \\ -\frac{1}{\lambda} & 0 & \dots & 0 & \frac{t_{0,0}^N t_{1,0}^0}{t_{1,0}^N t_{0,0}^0} \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \lambda \frac{t_{0,1}^1 t_{0,0}^N}{t_{0,0}^0 t_{0,1}^0} \\ \frac{t_{0,1}^2 t_{0,0}^1}{t_{1,0}^0 t_{0,1}^1} & 1 & 0 & \dots & 0 & 0 \\ 0 & \frac{t_{0,1}^3 t_{0,0}^2}{t_{0,0}^3 t_{0,1}^2} & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & \dots & 0 & \frac{t_{0,0}^{N-1} t_{0,1}^0}{t_{0,0}^N t_{0,1}^N} & 1 \end{pmatrix}.$$

If the functions  $t^0, t^1, \dots, t^N$  solve the system (7.3) then the overdetermined system of linear equations

$$P_{1,0} = UP, \quad P_{0,1} = VP \quad (7.4)$$

is compatible (see [32]).

**Remark 3.** Note that system (7.3) slightly differs from the system corresponding to the same algebra  $A_N^{(1)}$  but defined by the formula (1.3). For example the system (7.3) for  $N = 2$  is of the

form

$$\begin{aligned}
t_{0,0}^0 t_{1,1}^0 - t_{1,0}^0 t_{0,1}^0 &= t_{1,0}^2 t_{0,1}^1, \\
t_{0,0}^1 t_{1,1}^1 - t_{1,0}^1 t_{0,1}^1 &= t_{1,0}^0 t_{0,1}^2, \\
t_{0,0}^2 t_{1,1}^2 - t_{1,0}^2 t_{0,1}^2 &= t_{1,0}^1 t_{0,1}^0.
\end{aligned} \tag{7.5}$$

In terms of the variables  $u^0 = -\log t^0$ ,  $u^1 = -\log t^1$ ,  $u^2 = -\log t^2$  it looks like

$$\begin{aligned}
\Delta(u^0) &= e^{u_{1,0}^0 + u_{0,1}^0 - u_{0,1}^1 - u_{1,0}^2}, \\
\Delta(u^1) &= e^{-u_{1,0}^0 + u_{1,0}^1 + u_{0,1}^1 - u_{0,1}^2}, \\
\Delta(u^2) &= e^{-u_{0,1}^0 - u_{1,0}^1 + u_{1,0}^2 + u_{0,1}^0}.
\end{aligned} \tag{7.6}$$

while formula (1.3) gives the system

$$\begin{aligned}
\Delta(u^0) &= e^{u_{1,0}^0 + u_{0,1}^0 - u_{0,1}^1 - u_{1,0}^2}, \\
\Delta(u^1) &= e^{-u_{1,0}^0 + u_{1,0}^1 + u_{0,1}^1 - u_{0,1}^2}, \\
\Delta(u^2) &= e^{-u_{1,0}^0 - u_{1,0}^1 + u_{1,0}^2 + u_{0,1}^0}.
\end{aligned} \tag{7.7}$$

Let us give also the Lax pair for the system (7.5). Let us consider vector  $P = (\psi^0, \psi^1, \psi^2)^T$ .

Introduce  $3 \times 3$  matrices

$$U = \begin{pmatrix} \frac{t_{0,0}^0 t_{1,0}^1}{t_{1,0}^0 t_{0,0}^1} & -1 & 0 \\ 0 & \frac{t_{1,0}^2 t_{0,0}^1}{t_{0,0}^2 t_{1,0}^1} & -1 \\ -\frac{1}{\lambda} & 0 & \frac{t_{0,0}^2 t_{1,0}^0}{t_{1,0}^2 t_{0,0}^0} \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & \lambda \frac{t_{0,1}^1 t_{0,0}^2}{t_{0,0}^0 t_{0,1}^0} \\ \frac{t_{0,1}^2 t_{0,0}^0}{t_{1,0}^1 t_{0,1}^0} & 1 & 0 \\ 0 & \frac{t_{0,1}^0 t_{0,0}^1}{t_{0,0}^2 t_{0,1}^1} & 1 \end{pmatrix}.$$

If the functions  $t^0, t^1, t^2$  satisfy the system (7.5) then the equations

$$P_{1,0} = UP, \quad P_{0,1} = VP. \tag{7.8}$$

are compatible.

**Remark 4.** The Lax pair (7.4) can be rewritten in terms of the Cartan-Weyl basis of the algebra  $A_N^{(1)}$ :

$$\begin{aligned}
P_{1,0} &= (-\Lambda + e^{U_{1,0} - U_{0,0}}) P, \\
P_{0,1} &= (E + e^{U_{0,1}} \bar{\Lambda} e^{-U_{0,0}}) P
\end{aligned} \tag{7.9}$$

where  $\Lambda = \sum_{i=0}^N f_i$ ,  $\bar{\Lambda} = \sum_{i=0}^N e_i$ ,  $U = \sum_{i=0}^N u^i h_i$ ,  $e^{-u^i} = t^i$ ,

$$[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \tag{7.10}$$

$$[h_i, e_j] = A_{ij}e_j, \quad [h_i, f_j] = -A_{ij}f_j, \quad (7.11)$$

and  $A_{ij}$  are elements of the Cartan matrix of the algebra  $A_N^{(1)}$ .

## 8 Conclusions

A map is suggested converting any  $N \times N$  matrix  $A$  to a finite system of difference-difference equations of exponential type (see (1.3)). A hypothesis is formulated claiming that if  $A$  coincides with the Cartan matrix of a finite or affine Lie algebra then the corresponding system is integrable. The hypothesis is approved by numerous examples. The systems obtained are rather simple and elegant. They essentially differ from those studied earlier (see survey [16] and references therein). For instance, the system corresponding the algebra  $G_2$  given in [16] reads as:

$$\begin{aligned} T_m^{(1)}(u-1)T_m^{(1)}(u+1) &= T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) + T_{3m}^{(2)}(u), \\ T_{3m}^{(2)}\left(u - \frac{1}{3}\right)T_{3m}^{(2)}\left(u + \frac{1}{3}\right) &= T_{3m-1}^{(2)}(u)T_{3m+1}^{(2)}(u) + T_m^{(1)}\left(u - \frac{2}{3}\right)T_m^{(1)}(u)T_m^{(1)}\left(u + \frac{2}{3}\right), \\ T_{3m+1}^{(2)}\left(u - \frac{1}{3}\right)T_{3m+1}^{(2)}\left(u + \frac{1}{3}\right) &= T_{3m}^{(2)}(u)T_{3m+2}^{(2)}(u) + T_m^{(1)}\left(u - \frac{1}{3}\right)T_m^{(1)}\left(u + \frac{1}{3}\right)T_{m+1}^{(1)}(u), \\ T_{3m+2}^{(2)}\left(u - \frac{1}{3}\right)T_{3m+2}^{(2)}\left(u + \frac{1}{3}\right) &= T_{3m+1}^{(2)}(u)T_{3m+3}^{(2)}(u) + T_m^{(1)}(u)T_{m+1}^{(1)}\left(u - \frac{1}{3}\right)T_{m+1}^{(1)}\left(u + \frac{1}{3}\right), \end{aligned} \quad (8.1)$$

while our formula (1.3) generates  $G_2$  system which can be represented as follows:

$$\begin{aligned} t_{n,m}^1 t_{n+1,m+1}^1 - t_{n+1,m}^1 t_{n,m+1}^1 &= t_{n,m+1}^2, \\ t_{n,m}^2 t_{n+1,m+1}^2 - t_{n+1,m}^2 t_{n,m+1}^2 &= (t_{n+1,m}^1)^3. \end{aligned} \quad (8.2)$$

In a recent article by Kimura, Yamashita and Nakamura [31] a new application of conserved quantities of discrete-time integrable systems to numerical computations is suggested. The systems studied in the present paper might have applications in such kind numerical methods.

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